

PRED-HILBERTOV PROSTOR $C_0[a, b]$

NEKA JE PROSTOR $X = C_0[a, b]$ DEFINISAN NA SLEDEĆI NAČIN

$$X = C_0[a, b] = \left\{ f: [a, b] \rightarrow \mathbb{R} : \begin{array}{l} f \text{ JE DEO PO DEO NEPREKIDNA,} \\ \left(\text{IMA KONAČNO MNOGO PREKIDA} \right) \\ a \leq x_1 < x_2 < \dots < x_n \leq b \\ \text{POSTOJE LEVI I DESNI LIMESI U } x_i, \\ f(x_i) = \frac{1}{2} (f(x_{i-}) + f(x_{i+})) \end{array} \right\}$$

PROSTOR X JE REALAN VEKTORSKI PROSTOR

SA SABIRANJEM $(f+g)(x) = f(x) + g(x)$ I MNOŽENJA FUNKCIJE SKALAROM

$$(\lambda f)(x) = \lambda f(x).$$

DEFINIŠIMO PRESLIKAVANJE

$$\langle \cdot, \cdot \rangle : C_0[a, b] \times C_0[a, b] \rightarrow \mathbb{R} \text{ SA}$$

$$\langle f, g \rangle = \int_a^b f(x)g(x)dx, \quad f, g \in C_0[a, b].$$

OVO PRESLIKAVANJE JE SKALARNI PROIZVOD

$$1) \langle f, g \rangle = \int_a^b f(x)g(x)dx = \int_a^b g(x)f(x)dx = \langle g, f \rangle \text{ ZA SVAKO } f, g \in X$$

$$2) \langle \lambda f, g \rangle = \int_a^b (\lambda f)(x)g(x)dx = \int_a^b \lambda f(x)g(x)dx = \int_a^b f(x)(\lambda g)(x)dx = \langle f, \lambda g \rangle$$

ZA SVAKO $f, g \in X$

$$3) \langle f+g, h \rangle = \int_a^b (f+g)(x)h(x)dx = \int_a^b (f(x)+g(x))h(x)dx =$$

$$= \int_a^b (f(x)h(x) + g(x)h(x))dx = \int_a^b f(x)h(x)dx + \int_a^b g(x)h(x)dx$$

$$= \langle f, h \rangle + \langle g, h \rangle \text{ ZA SVAKO } f, g, h \in X$$

$$4) \langle f, f \rangle = \int_a^b f(x) \cdot f(x)dx = \int_a^b f^2(x)dx \geq 0 \text{ ZA SVAKO } f \in X$$

$$5) \text{ AKO JE } f \equiv 0 \text{ NA } [a, b], \text{ Onda JE } \langle f, f \rangle = \int_a^b f^2(x)dx = \int_a^b 0dx = 0$$

$$\text{NEKA JE } \langle f, f \rangle = 0, \text{ TO JEST } \int_a^b f^2(x)dx = 0.$$

ODAVDE $\int_{x_{k-1}}^{x_k} f^2(x) dx = 0$ ZA SVAKO $k=1, 2, \dots, n$.

MOŽEMO DEFINISATI FUNKCIJE $f_k: [x_{k-1}, x_k] \rightarrow \mathbb{R}$

$$f_k(x) = \begin{cases} f(x), & x \in [x_{k-1}, x_k] \\ f(x_{k-1+}), & x = x_{k-1} \\ f(x_{k-}) & , x = x_k \end{cases}$$

KOJE SU NEPREKIDNE NA $[x_{k-1}, x_k]$ I VAŽI $\int_{x_{k-1}}^{x_k} f_k^2(x) dx = 0$,

(RAZLIKUJU SE OD FUNKCIJE f SAMO U KRAJNIM TAČKAMA, PA SU INTEGRALI JEDNAKI, TO JEST $\int_{x_{k-1}}^{x_k} f_k^2(x) dx = \int_{x_{k-1}}^{x_k} f^2(x) dx = 0$)

ODAKLE JE $f_k \equiv 0$ NA $[x_{k-1}, x_k]$, ODHODNO $f \equiv 0$ NA (x_{k-1}, x_k) ZA SVAKO k .

Iz $f(x_{k+}) = \lim_{x \rightarrow x_{k+}} f(x) = 0$ i $f(x_{k-}) = \lim_{x \rightarrow x_{k-}} f(x) = 0$, SLEDI DA JE

$$f(x_k) = \frac{1}{2} (f(x_{k-}) + f(x_{k+})) = \frac{1}{2} (0 + 0) = 0.$$

DAKLE, $f \equiv 0$ NA $[a, b]$.

DOKAZALI SMO DA JE PRESLIKAVANJE $\langle \cdot, \cdot \rangle$ SKALARNI PROIZVOD

NAPOMENA USLOV $f(x_k) = \frac{1}{2} (f(x_{k-}) + f(x_{k+}))$ NAM JE BIO POTREBAN KOD PROSTORA $C_0[a, b]$ DA BI VAŽILA IMPLIKACIJA $\langle f, f \rangle = 0 \Rightarrow f = 0$

DEFINISALI SMO PRED-HILBERTOV PROSTOR $C_0[a, b]$ SA SKALARNIM

PROIZVODOM $\langle f, g \rangle = \int_a^b f(x)g(x) dx$ I NORMOM $\|f\| = \sqrt{\langle f, f \rangle} = \sqrt{\int_a^b f^2(x) dx}$.

VAŽI KOŠI-ŠVARCOVA NEJEDNAKOST $|\int_a^b f(x)g(x) dx| \leq \sqrt{\int_a^b f^2(x) dx} \cdot \sqrt{\int_a^b g^2(x) dx}$

I NEJEDNAKOST MINIKOVSKOG

$$\sqrt{\int_a^b (f(x) + g(x))^2 dx} \leq \sqrt{\int_a^b f^2(x) dx} + \sqrt{\int_a^b g^2(x) dx}$$

NIZ (f_n) KONVERGIRA (U SREDNJEM, U NORMI) KA $f \in C_0[a, b]$ AKO

$$\lim_{n \rightarrow \infty} \|f_n - f\| = 0, \text{ T.J. } \lim_{n \rightarrow \infty} \int_a^b (f_n(x) - f(x))^2 dx = 0.$$

STAV U PRED-HILBERTOVOM PROSTORU $C_0[-l, l]$ ($l > 0$) SISTEM VEKTORA

$$(1) \quad \frac{1}{\sqrt{2l}}, \frac{1}{\sqrt{l}} \cos \frac{\pi x}{l}, \frac{1}{\sqrt{l}} \sin \frac{\pi x}{l}, \frac{1}{\sqrt{l}} \cos \frac{2\pi x}{l}, \frac{1}{\sqrt{l}} \sin \frac{2\pi x}{l}, \dots, \frac{1}{\sqrt{l}} \cos \frac{n\pi x}{l}, \dots$$

JE ORTONORMIRAN.

MOŽE SE POKAZATI

$$\int_{-e}^e \frac{1}{\sqrt{e}} \cos \frac{m\pi x}{e} \cdot \frac{1}{\sqrt{e}} \cos \frac{n\pi x}{e} dx = \begin{cases} 1, & m=n \in \mathbb{N} \\ 0, & m \neq n, m, n \in \mathbb{N} \end{cases}$$

$$\int_{-e}^e \frac{1}{\sqrt{e}} \sin \frac{m\pi x}{e} \cos \frac{n\pi x}{e} dx = 0 \quad \text{ZA } m, n \in \mathbb{N}$$

$$\int_{-e}^e \frac{1}{\sqrt{e}} \sin \frac{m\pi x}{e} \sin \frac{n\pi x}{e} dx = \begin{cases} 1, & m=n \in \mathbb{N} \\ 0, & m \neq n, m, n \in \mathbb{N} \end{cases}$$

$$\int_{-e}^e \frac{1}{\sqrt{2e}} \cdot \frac{1}{\sqrt{e}} \cos \frac{n\pi x}{e} dx = 0, \quad \int_{-e}^e \frac{1}{\sqrt{2e}} \cdot \frac{1}{\sqrt{e}} \sin \frac{n\pi x}{e} dx = 0, \quad \int_{-e}^e \frac{1}{\sqrt{2e}} \cdot \frac{1}{\sqrt{2e}} dx = 1$$

NA PRIMER

$$\begin{aligned} \bar{I} &= \int_{-e}^e \frac{1}{\sqrt{e}} \cos \frac{m\pi x}{e} \cdot \frac{1}{\sqrt{e}} \cos \frac{n\pi x}{e} dx = \frac{1}{e} \int_{-e}^e \cos \frac{m\pi x}{e} \cos \frac{n\pi x}{e} dx = \\ & \qquad \qquad \qquad \cos \alpha \cos \beta = \frac{1}{2} (\cos(\alpha - \beta) + \cos(\alpha + \beta)) \\ &= \frac{1}{e} \int_{-e}^e \frac{1}{2} (\cos \frac{\pi x}{e} (m-n) - \cos \frac{\pi x}{e} (m+n)) dx \end{aligned}$$

$$\begin{aligned} \text{AKO JE } m=n, \text{ ODLIČNO JE } \bar{I} &= \frac{1}{e} \int_{-e}^e \frac{1}{2} (1 - \cos \frac{\pi x}{e} \cdot 2n) \\ &= \frac{1}{2e} \left(x \Big|_{-e}^e - \frac{e}{2n\pi} \sin \frac{\pi x}{e} 2n \Big|_{-e}^e \right) \\ &= \frac{1}{2e} (2e - 0) = 1 \end{aligned}$$

AKO JE $m \neq n$, ODLIČNO JE

$$\begin{aligned} \bar{I} &= \frac{1}{2e} \int_{-e}^e (\cos \frac{\pi x}{e} (m-n) - \cos \frac{\pi x}{e} (m+n)) dx = \\ &= \frac{1}{2e} \left(\frac{e}{\pi(m-n)} \sin \frac{\pi x}{e} (m-n) \Big|_{-e}^e - \frac{e}{\pi(m+n)} \sin \frac{\pi x}{e} (m+n) \Big|_{-e}^e \right) \\ &= \frac{1}{2e} (0 - 0) = 0 \end{aligned}$$

U $C_0[-\pi, \pi]$ IMAMO ONS

$$\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos x, \frac{1}{\sqrt{\pi}} \sin x, \frac{1}{\sqrt{\pi}} \cos 2x, \frac{1}{\sqrt{\pi}} \sin 2x, \dots, \frac{1}{\sqrt{\pi}} \cos nx, \frac{1}{\sqrt{\pi}} \sin nx, \dots$$

FURIJEONI KOEFICIJENTI FUNKCIJE $f \in C_0[-l, l]$ U ODHOSU NA SISTEM (1)

$$a_0 = \langle f, e_0 \rangle = \int_{-l}^l f(x) \cdot \frac{1}{\sqrt{2l}} dx = \frac{1}{\sqrt{2l}} \int_{-l}^l f(x) dx$$

$$a_0 e_0 = \frac{1}{\sqrt{2l}} \int_{-l}^l f(x) dx \cdot \frac{1}{\sqrt{2l}} = \frac{1}{2l} \int_{-l}^l f(x) dx$$

$$a_n = \langle f, \frac{1}{\sqrt{l}} \cos \frac{n\pi x}{l} \rangle = \int_{-l}^l f(x) \cdot \frac{1}{\sqrt{l}} \cos \frac{n\pi x}{l} dx = \frac{1}{\sqrt{l}} \int_{-l}^l f(x) \cos \frac{n\pi x}{l} dx$$

$$a_n \frac{1}{\sqrt{l}} \cos \frac{n\pi x}{l} = \frac{1}{l} \int_{-l}^l f(x) \cos \frac{n\pi x}{l} dx \quad n=1, 2, \dots$$

$$b_n = \langle f, \frac{1}{\sqrt{l}} \sin \frac{n\pi x}{l} \rangle = \frac{1}{\sqrt{l}} \int_{-l}^l f(x) \sin \frac{n\pi x}{l} dx$$

$$b_n \frac{1}{\sqrt{l}} \sin \frac{n\pi x}{l} = \frac{1}{l} \int_{-l}^l f(x) \sin \frac{n\pi x}{l} dx$$

ZAMENIMO KOEFICIJENTE a_0, a_n I b_n SA

$$a_0 = \frac{1}{l} \int_{-l}^l f(x) dx, \quad a_n = \frac{1}{l} \int_{-l}^l f(x) \cos \frac{n\pi x}{l} dx, \quad b_n = \frac{1}{l} \int_{-l}^l f(x) \sin \frac{n\pi x}{l} dx$$

(ZVACEMO IH TRIGONOMETRIJSKI FURIJEONI KOEFICIJENTI)

FURIJEON RED (ZVACEMO GA TRIGONOMETRIJSKIM) FUNKCIJE f JE

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l}$$

U SLUCAJU $C_0[-\pi, \pi]$ ($l=\pi$) TRIGONOMETRIJSKI FURIJEON RED JE

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

GDE SU FURIJEONI KOEFICIJENTI

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx, \quad n \in \mathbb{N}$$

TEOREMA TRIGONOMETRIJSKI SISTEM JE POTPUN U $C_0[-\ell, \ell]$, $\ell > 0$.

POSLEDICA a) TRIGONOMETRIJSKI FURIJEOV RED FUNKCIJE KONVERGIRA (U SREDNJEM, U NORMI) KA FUNKCIJI, TO JEST

$$\lim_{m \rightarrow \infty} \int_{-\ell}^{\ell} \left(f(x) - \left(\frac{a_0}{2} + \sum_{n=1}^m a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell} \right) \right)^2 dx = 0$$

$$\text{ili } \|f - T_m\|_2 \rightarrow 0 \quad (m \rightarrow \infty) \text{ GDE JE } T_m(x) = \frac{a_0}{2} + \sum_{n=1}^m a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell}$$

b) VAŽI PARSEVALOVA JEDNAKOST

$$\frac{1}{\ell} \int_{-\ell}^{\ell} f^2(x) dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

c) AKO FUNKCIJE $f, g \in C_0[-\ell, \ell]$, $\ell > 0$ IMAJU ISTE FURIJEOVE KOEFICIJENTE ONDA JE $f \equiv g$.

AKO JE FUNKCIJA $f \in C_0[-\ell, \ell]$ PARNA, ONDA JE $b_n = 0$ ZA SVAKO $n \in \mathbb{N}$.

AKO JE FUNKCIJA $f \in C_0[-\ell, \ell]$ NEPARNA, ONDA JE $a_n = 0$ ZA SVAKO $n \in \mathbb{N}_0$.

PREMA TOME, ZA PARNU FUNKCIJU DOBIJEMO KOSINUSNI FURIJEOV RED, A ZA NEPARNU FUNKCIJU SINUSNI FURIJEOV RED.

PRIMER NEKA JE $f: [-\pi, \pi] \rightarrow \mathbb{R}$, $f(x) = x$. FUNKCIJA f PRIPADA PROSTORU $C_0[-\pi, \pi]$,

PA SU FURIJEVI KOEFICIJENTI JEDNAKI:

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = 0 \quad (\text{JER JE } f \text{ NEPARNA}),$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = 0 \quad (\text{JER JE } f(x) \cos x \text{ NEPARNA})$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{x \sin nx}_{\text{PARNA}} dx = \frac{2}{\pi} \int_0^{\pi} x \sin nx dx$$

$$\begin{aligned} &= \left(\begin{array}{l} u = x \Rightarrow du = dx \\ dv = \sin nx dx \Rightarrow v = -\frac{1}{n} \cos nx \end{array} \right) = -\frac{2}{n\pi} x \cos nx \Big|_0^{\pi} + \frac{2}{n\pi} \int_0^{\pi} \cos nx dx \\ &= -\frac{2}{n} \cos(n\pi) + \frac{2}{n\pi} \cdot 0 \cos n \cdot 0 + \frac{2}{n^2\pi} \cdot \sin nx \Big|_0^{\pi} \end{aligned}$$

$$= -\frac{2}{n}(-1)^n + 0 + \frac{2}{n^2\pi}(\sin n\pi - \sin n \cdot 0) = \frac{2}{n}(-1)^{n+1}$$

TRIGONOMETRIJSKI FURIJEV RED FUNKCIJE $f \in C_0[-\pi, \pi]$ IMA OBLIK

$$\sum_{n=1}^{\infty} b_n \sin nx = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin nx$$

PREMA POSLEDICI a) FURIJEV RED FUNKCIJE f KONVERGIRA U
SREDNJEM, TO JEST

$$\lim_{n \rightarrow \infty} \int_{-\pi}^{\pi} \left(x - \sum_{n=1}^{\infty} \frac{2 \cdot (-1)^{n+1}}{n} \sin nx \right)^2 dx = 0$$

PREMA POSLEDICI b) VAŽI PARSEVALOVA JEDNAKOST

$$\sum_{n=1}^{\infty} \left(\frac{2}{n}(-1)^{n+1} \right)^2 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx,$$

$$\text{ODAKLE DOBIVAMO } 4 \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{\pi} \frac{x^3}{3} \Big|_{-\pi}^{\pi} = \frac{2\pi^2}{3}, \text{ ODNOSHO } \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

FURIJEV RED $\sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin nx$ NE KONVERGIRA TAČKA PO TAČKA (T.J. OBČNO

KA FUNKCIJI $f(x) = x$ NA $[-\pi, \pi]$, JER JE $f(\pi) = \pi$ I $\sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin n\pi = 0$

NEKA JE $f: \mathbb{R} \rightarrow \mathbb{R}$ 2π PERIODIČNA I APSOLUTNO INTEGRABILNA NA $[-\pi, \pi]$

FURIJEOV RED FUNKCIJE (AKO SU ZADOVOLJEKI USLOVI IZ $C_0[-\pi, \pi]$) JE

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx, \text{ GDE SU } a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx, n \in \mathbb{N}_0$$

PARCIJALNA SUMA m -TOG REDA JE

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx, n \in \mathbb{N}$$

$$S_m(x) = \frac{a_0}{2} + \sum_{n=1}^m a_n \cos nx + b_n \sin nx,$$

$$\text{ODAKLE JE } S_m(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt + \sum_{n=1}^m \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nx dt \cdot \cos nx + \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt dt \cdot \sin nx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \left[1 + 2 \sum_{n=1}^m \cos nt \cos nx - \sin nt \sin nx \right] dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \left[1 + 2 \sum_{n=1}^m \cos n(x-t) \right] dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \underbrace{\frac{\sin(m+\frac{1}{2})(x-t)}{\sin \frac{1}{2}(x-t)}}_{D_m(x-t)} dt$$

$$D_m(u) = \frac{\sin(m+\frac{1}{2})u}{\sin \frac{1}{2}u} \quad \text{DIRIHLEOVO JEZGRO}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) D_m(x-t) dt.$$

⊙ JEDNAKOST VAŽI NA OSNOVU JEDNAKOSTI

$$1 + 2 \sum_{n=1}^m \cos nu = \frac{\sin(m+\frac{1}{2})u}{\sin \frac{1}{2}u} = D_m(u)$$

DIRIHLEOVO JEZGRO JE 2π PERIODIČNA I PARNA FUNKCIJA I VAŽI

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} D_m(u) du &= \frac{1}{\pi} \int_0^{\pi} D_m(u) du = \frac{1}{\pi} \int_0^{\pi} \left(1 + 2 \sum_{n=1}^m \cos nu \right) du = \frac{1}{\pi} \left(u \Big|_0^{\pi} + 2 \sum_{n=1}^m \frac{1}{n} \sin nu \Big|_0^{\pi} \right) \\ &= \frac{1}{\pi} \left(\pi + 2 \sum_{n=1}^m \frac{1}{n} (0-0) \right) = 1 \end{aligned}$$

LEMA (RIMANOVA)

AKO JE $f: (a, b) \rightarrow \mathbb{R}$ APSOLUTNO INTEGRABILNA, ONDA JE

$$\lim_{\lambda \rightarrow \infty} \int_a^b f(x) \cos \lambda x dx = \lim_{\lambda \rightarrow \infty} \int_a^b f(x) \sin \lambda x dx = 0$$