③ $f(x) = \dfrac{1+\ln|x|}{x(1-\ln|x|)}$

① Domen $\quad x \in \mathbb{R} \setminus \{0, -e, e\}$

② Nule i znaci

$f(x) = 0 \iff 1+\ln|x| = 0 \iff x = \pm 1/e$

	$-\infty$ do $-e$	$-e$ do $-1/e$	$-1/e$ do 0	0 do $1/e$	$1/e$ do e	e do $+\infty$
$1+\ln\lvert x\rvert$	+	+	−	−	+	+
$1-\ln\lvert x\rvert$	−	+	+	+	+	−
x	−	−	−	+	+	+
$f(x)$	+	−	+	−	+	−

③ Parnost/Neparnost periodičnost $\to$ nije

$f(-x) = \dfrac{1+\ln|x|}{-x(1-\ln|x|)} = -f(x) \Rightarrow f$ neparna Ispitujemo samo $(0,e)\cup(e,+\infty)$

4. Asimptote

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{1+\overbrace{\ln x}^{\to -\infty}}{x(1-\ln x)} = \lim_{x\to 0^+} \frac{1}{x}\cdot\frac{1+\ln x}{1-\ln x} = \lim_{x\to 0^+} \overbrace{\frac{1}{x}}^{\to +\infty}\Big(-1+\overbrace{\frac{2}{\underbrace{1-\ln x}_{\to -\infty}}}^{\to 0}\Big) = -\infty$$

$$\lim_{x\to e^-} f(x) = \lim_{x\to e^-} \underbrace{\frac{1}{x}}_{\to 1/e}\Bigg(\frac{\overbrace{1+\ln x}^{\to 2}}{\underbrace{1-\ln x}_{\to 0^+}}\Bigg) = +\infty$$

$$\lim_{x\to e^+} f(x) = \lim_{x\to e^+} \underbrace{\frac{1}{x}}_{\to 1/e}\Bigg(\frac{\overbrace{1+\ln x}^{\to 2}}{\underbrace{1-\ln x}_{\to -0}}\Bigg) = -\infty$$

Kose/Horizontalne

$$\lim_{x\to+\infty} \frac{1+\ln x}{(1-\ln x)x} = \lim_{x\to+\infty} \underbrace{\frac{1}{x}}_{\to 0}\underbrace{\Big(-1+\overbrace{\frac{2}{1-\ln x}}^{\to 0}\Big)}_{\to -1} = 0 \Rightarrow$$ Za $x\to+\infty$ imamo horizontalnu $y=0$

$$a = \lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \frac{1}{x^2}\Big(-1+\frac{2}{1-\ln x}\Big) = 0$$

$$b = \lim_{x\to+\infty} (f(x) - 0x) = \lim_{x\to+\infty} = 0 \longrightarrow y = 0\cdot x + 0 \Rightarrow y = 0$$

⑤ Monotonost

$x>0 \quad f(x) = \dfrac{1+\ln x}{x(1-\ln x)}$

$$f'(x) = \frac{1/x \cdot x(1-\ln x) - (1+\ln x)(1 - x\cdot 1/x - \ln x)}{x^2(1-\ln x)^2}$$

$$f'(x) = \frac{1-\ln x+\ln x+\ln^2 x}{x^2(1-\ln x)^2} = \frac{\ln^2 x+1}{x^2(1-\ln x)^2} > 0$$ $f\uparrow$ na $x>0$

⑥ Konveksnost

$$f''(x) = 2\ln x \cdot \frac{1}{x} \cdot \frac{1}{x^2(1-\ln x)^2} - (1+\ln^2 x)\frac{2(-\ln x)}{x^3(1-\ln x)^3}$$

$$= \frac{2\ln x}{x^3(1-\ln x)^2} - (1+\ln^2 x)\frac{2(-\ln x)}{x^3(1-\ln x)^3} = \frac{2\ln x(1-\ln x) + 2\ln x(1+\ln^2 x)}{x^3(1-\ln x)^3}$$

$$= \frac{2\ln x(\underbrace{\ln^2 x - \ln x + 2}_{>0})}{\underbrace{x^3}_{x>0,\ x^3>0}(1-\ln x)^3}$$

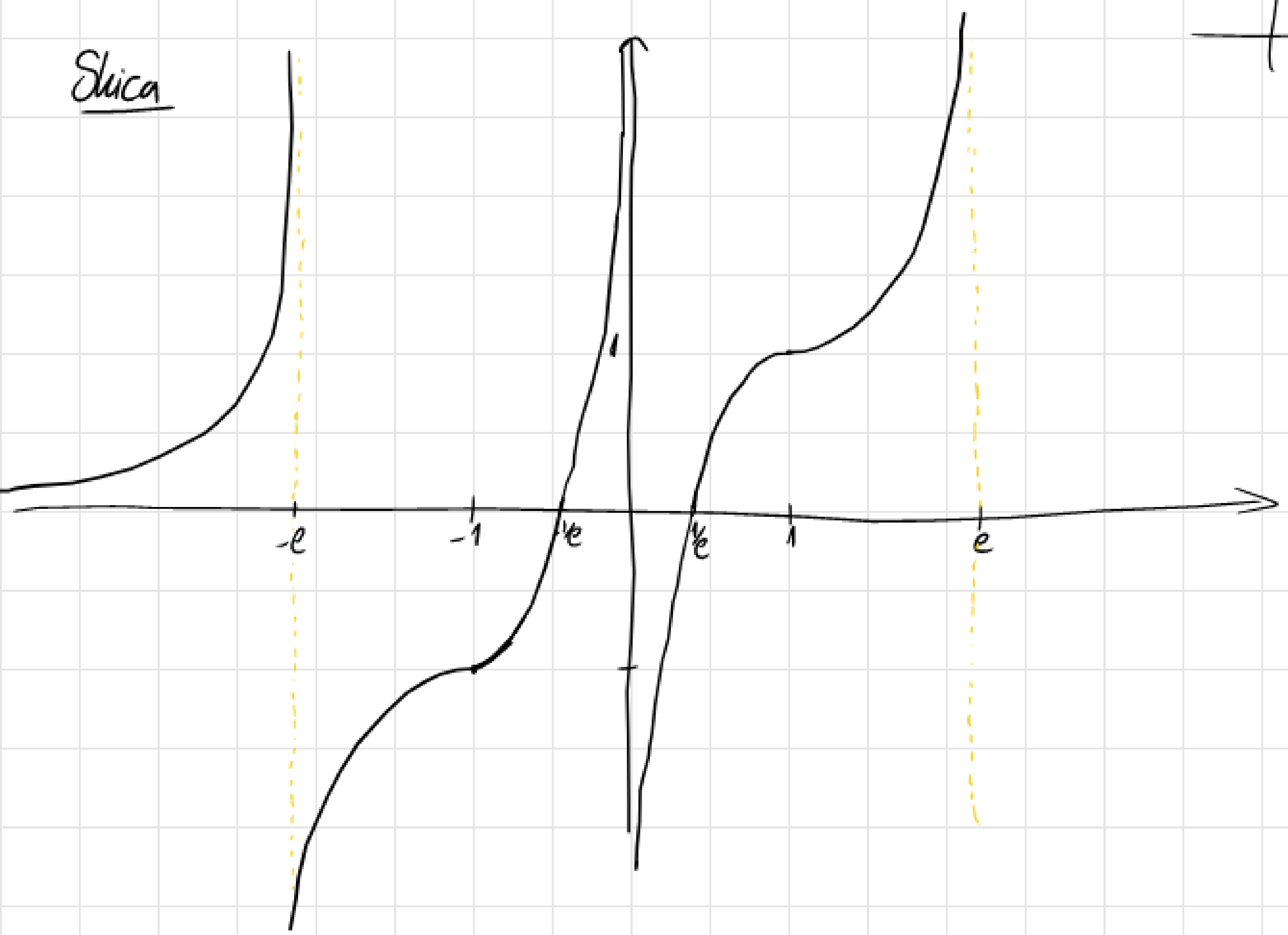
$$\operatorname{sgn}(f''(x)) = \operatorname{sgn}\left(\frac{2\ln x}{(1-\ln x)^3}\right)$$

	0	1	e	+∞
$\ln x$	$-$	$+$	$+$	
$1-\ln x$	$+$	$+$	$-$	
	$-$	$+$	$-$	
	$\cap$	$\cup$	$\cap$	

prevojna

nije def, pa nije

Slika



Nizovi

$a : \mathbb{N} \to \mathbb{R} \qquad a(n) \leftrightarrow a_n$

$$\lim_{n\to\infty} a_n = a \iff (\forall \varepsilon > 0)(\exists n_0 \in \mathbb{N})\ n \geq n_0 \Rightarrow |a_n - a| < \varepsilon$$

Osobine: $\lim\limits_{n\to\infty} a_n = a \in \mathbb{R} \qquad \lim\limits_{n\to\infty} b_n = b \in \mathbb{R}$

$$\lim_{n\to\infty}(a_n \pm b_n) = a \pm b \qquad \lim_{n\to\infty}(a_n \cdot b_n) = a \cdot b \qquad b \neq 0\ \lim_{n\to\infty}\frac{a_n}{b_n} = \frac{a}{b}$$

* $a_n = \frac{2n+1}{n-2} \qquad \lim\limits_{n\to\infty} a_n = 2$

$\varepsilon > 0$ proizvoljno $\qquad$ Tražimo $n_0 \in \mathbb{N}$ t.d. za $n \geq n_0$ $\left|\frac{2n+1}{n-2} - 2\right| < \varepsilon$

$$\left|\frac{2n+1-2n+4}{n-2}\right| = \left|\frac{5}{n-2}\right| \leq \varepsilon \qquad \frac{5}{n_0-2} \leq \varepsilon \qquad \frac{5}{\varepsilon} + 2 \leq n_0$$

$$n_0 = \left\lfloor \frac{5}{\varepsilon} \right\rfloor + 3 \Rightarrow \text{za } n > n_0 \text{ važi } \frac{5}{n-2} < \frac{5}{n_0-2} < \varepsilon$$

Daule $(\forall \varepsilon>0)(\exists n_0 \in \mathbb{N})$ $n>n_0 \Rightarrow |a_n-2|<\varepsilon$

$\lfloor \frac{5}{\varepsilon} \rfloor + 3$

Def $n>n_0 \Rightarrow |a_n-2|<\varepsilon$ $\lim\limits_{n\to\infty} a_n = 2$

* $q\in\mathbb{R}$ $\lim\limits_{n\to\infty} q^n = \begin{cases} 0, & |q|<1 \\ 1, & q=1 \\ \infty, & q>1 \\ \text{ne postoji}, & \text{inače} \end{cases}$

1) $k\in\mathbb{N}$

$$\lim_{n\to\infty} \frac{\binom{n}{k}}{n^k} = \lim_{n\to\infty} \frac{\frac{n(n-1)\dots(n-k+1)}{k!}}{n^k} = \lim_{n\to\infty} \frac{1}{k!}\frac{n}{n}\frac{(n-1)}{n}\frac{(n-2)}{n}\dots\frac{n-k+1}{n} = 1$$

$(\forall_i \varepsilon$

2 $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$

$$\lim_{n\to\infty} \frac{1^2+3^2+\dots+(2n-1)^2}{n^3} = ?$$

$$1^2+3^2+\dots+(2n-1)^2 = (1^2+2^2+3^2+\dots+(2n)^2-(2^2+\dots+(2n)^2) =$$

$$= \frac{2n(2n+1)(4n+1)}{6} - 4\frac{n(n+1)(2n+1)}{6} = 1/3\left(n(2n+1)(4n+1) - 2n(n+1)(2n+1)\right)$$

$$= 1/3\,(8n^3+6n^2+n-4n^3-6n^2-2n) = 1/3(4n^3-n)$$

$$* = \lim_{n\to\infty} \frac{1/3(4n^3-n)}{n^3} = \lim_{n\to\infty}\left(4/3 - \underbrace{\frac{1}{3n^2}}_{\to 0}\right) = 4/3$$

$$= \lim_{n\to\infty} \frac{2n(2n+1)(4n+1)}{n\cdot n\cdot 6n} - \frac{4n(n+1)(2n+1)}{6n\cdot n\cdot n} = \frac{2\cdot 2\cdot 4}{6} - \frac{4\cdot 2}{6} = 4/3$$

3 $$\lim_{n\to\infty}\sqrt{n^2+n+1}-\sqrt{n^2-n+1} = \lim_{n\to\infty}\frac{n^2+n+1-n^2+n-1}{\sqrt{n^2+n+1}+\sqrt{n^2-n+1}} = \lim \frac{2n}{\sqrt{n^2+n+1}+\sqrt{n^2-n+1}} =$$

$$= \lim_{n\to\infty}\frac{2\cdot n}{n\left(\sqrt{1+1/n+1/n^2}+\sqrt{1-1/n+1/n^2}\right)} = 2\cdot\frac{1}{1+1} = 1$$

4 $\lim\limits_{n\to\infty}\left(\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\dots+\frac{1}{n(n+1)}\right) = ?$ $\frac{1}{k(k+1)} = 1/k - \frac{1}{k+1}$

$$a_n = 1/1\cdot2 + 1/2\cdot3 + \dots + \frac{1}{n(n+1)} = 1-\cancel{1/2}+\cancel{1/2}-\cancel{1/3}+\cancel{1/3}+\dots+\cancel{1/n}-\frac{1}{n+1} = 1-\frac{1}{n+1}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1-\frac{1}{n+1}\right) = 1$$

T o 3 limesa (sendvič/2 policajca)

$$a_n \le b_n \le c_n$$
$$\searrow \quad \downarrow \quad \swarrow$$
$$x$$

$\ast$ $\lim \sqrt[n]{x_1^n + x_2^n + \dots + x_k^n} = \max\{x_1, \dots, x_k\}$ $\quad x_i = \max\{x_1, \dots, x_k\}$

$$x_i \le \lim_{n\to\infty} x_i \sqrt[n]{\underbrace{\left(\frac{x_1}{x_i}\right)^n + \dots + 1 + \dots + \left(\frac{x_k}{x_i}\right)^n}_{\le k}} \le \lim_{n\to\infty} x_i \sqrt[n]{k \cdot 1}$$
$$\to 1 \qquad\qquad 1 \qquad\qquad 1$$

4. $\lim_{n\to\infty} \sqrt[n]{2^n + 500^n + (\sqrt{3})^n} = 500$

5. $\lim_{n\to\infty} \sqrt[n]{1 + x^{2n} + x^{5n}} = \max\{1, x^{2n}, x^{5n}\}$ $\quad x>0$ u zavisnosti od x

$x > 1 \to \max\{1, x^2, x^5\} = x^5$

$x \le 1 \to \max\{1, x^2, x^5\} = 1$

Domaći

$\lim_{n\to\infty} \sqrt[n]{\frac{1}{x^{5n}} + x^n + x^{5n}}$

5n piše

6. $\lim_{n\to\infty} \left(\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \dots + \frac{1}{\sqrt{n^2+n}}\right)$

$$\sqrt{\frac{n^2}{n^2+n}} = \frac{n}{\sqrt{n^2+n}} \le \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \dots + \frac{1}{\sqrt{n^2+n}} \le \frac{n}{\sqrt{n^2+1}} = \sqrt{\frac{n^2}{n^2+1}} = \sqrt{\frac{1}{1+1/n^2}}$$

$$\sqrt{\frac{1}{1+1/n}} \to 1 \qquad \to 1 \leftarrow \qquad 1$$

1 Ako je niz monoton i ograničen, onda je konv.

$a_n \nearrow$ Dovoljno je da bude ogr. odozgo

$a_n \searrow$ — || — odozdo

$a_n \nearrow$ ako $a_{n+1} \ge a_n$ $\qquad a_n \searrow$ ako $a_{n+1} \le a_n$

$f_{min} = \min f(x), \; x \in [a,b]$ $\qquad f_{max} = \max f(x), \; x \in [a,b]$

$f([a,b]) = [f_{min}, f_{max}]$

7. Ispitati konv.

$$X_n = \underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}}_{n \text{ korenova}}$$

Monotonost $\quad X_{n+1} = \sqrt{2+X_n}$

$(X_{n+1})^2 = 2+X_n$

$$\sqrt{2+X_n} \overset{?}{\geq} X_n \iff 2+X_n \geq X_n^2 \iff 2+X_n-X_n^2 \geq 0$$

$X_n \in (-1, 2)$ za $x_n > 2 \quad X_{n+1} \leq X_n$

$x_n \leq 2 \quad x_{n+1} \geq x_n$

$x_3 \leq 2 \iff x_n \geq x_3$

Indukcija: Baza $x_1 = \sqrt{2} < 2$

korak: $x_n < 2 \implies \sqrt{2+x_n} < 2$?

$\iff 2+X_n < 4 \quad X_n < 2$

$x_n < 2$

$x_n + 2 < 2^2$

$\sqrt{x_n+2} < 2$

$\implies (\forall n \in \mathbb{N})\ x_n < 2$

$\implies x_{n+1} > x_n \to x_n \nearrow \quad x_n$ odozgo (?)

$\exists \lim_{n\to\infty} x_n = L \in \mathbb{R}$

$$L = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \sqrt{x_n+2} = \sqrt{L+2}$$

$L^2 = L+2 \quad L^2 - L - 2 = 0 \quad \left.\begin{matrix}(L+1)(L-2) = 0 \\ L > 0\end{matrix}\right\} L = 2$

$\implies x_n$ konv.

Objašnjenje:

Hoću da dokažem da je rastuća $\sqrt{x_n+2} \geq X_n \iff 2+X_n \geq X_n^2$

$\iff x_n^2 - x - 2 \leq 0 \iff (x_n+1)(x_n-2) \leq 0 \iff x_n \leq 2$

$x_{n+1} \geq x_n \iff x_n \leq 2$

Ako je $X_n \leq 2$ onda je $x_{n+1} \geq x_n \quad X_{n+1} \leq X_n$

Indukcijom dokaz da $(\forall n \in \mathbb{N})\ x_n \leq 2 \implies (\forall n \in \mathbb{N})\ x_{n+1} \geq x_n$

$\to x_n \nearrow$ rastuća

$(\forall n \in \mathbb{N})\ x_n \leq 2$

$\left.\begin{matrix}\text{rastući} \\ \text{ograničen}\end{matrix}\right\} \implies$ konv.

8. $a > 0 \quad x_1 > 0$ dato

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

Ispitati konv.

$$L = \frac{1}{2}\left(L + \frac{a}{L}\right)$$

$$\tfrac{1}{2}L = \tfrac{1}{2}\frac{a}{L}$$
$$L^2 = a \quad \rightarrow L = \sqrt{a}$$
$$x_{n+1} \overset{?}{\geq} x_n$$
$$\tfrac{1}{2}\left(x_n + \frac{a}{x_n}\right) \geq x_n$$
$$\cancel{\tfrac{1}{2}} \cdot \frac{a}{x_n} \geq \cancel{\tfrac{1}{2}} x_n \quad , x_n > 0$$
$$a \geq x_n^2$$
$$x_n \leq \sqrt{a}$$

$$x_{n+1} \geq x_n \iff x_n \leq \sqrt{a}$$
$$x_n \leq \sqrt{a} \Rightarrow x_{n+1} \geq x_n$$
$$x_n \geq \sqrt{a} \Rightarrow x_{n+1} \leq x_n$$

x_1 može da bude bilo kakav

$$x_2 = \tfrac{1}{2}\left(x_1 + \frac{a}{x_1}\right) \ \Box \ \sqrt{a}$$
$$x_2 \geq \sqrt{x_1 \cdot \frac{a}{x_1}} = \sqrt{a}$$
$$n \geq 2 \quad x_n = \tfrac{1}{2}\left(x_{n-1} + \frac{a}{x_{n-1}}\right) \overset{AG}{\geq} \sqrt{x_{n-1} \cdot \frac{a}{x_{n-1}}} = \sqrt{a}$$
$$\Rightarrow (\forall n \geq 2)\ x_n \geq \sqrt{a} \rightarrow x_{n-n} \leq x_n$$

$$\left.\begin{array}{l} x_n \downarrow \text{ opadajući} \\ x_n \text{ ograničen odozdo} \end{array}\right\} x_n \text{ konv.}$$

$$\lim_{n\to\infty} x_n = L$$
$$x_{n+1} = \tfrac{1}{2}\left(x_n + \frac{a}{x_n}\right) \qquad \lim x_n = \lim \tfrac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$
$$L = \tfrac{1}{2}\left(L + \frac{a}{L}\right) - L = \frac{a}{L} \quad \left.\begin{array}{l} L^2 = a \\ L > 0 \end{array}\right\} L = \sqrt{a}$$

9) $a_n = \dfrac{c^n}{n!}$, $c > 0$

$$a_{n+1} = \frac{c \cdot c^n}{(n+1)\,n!} = \frac{c}{n+1} \cdot a_n \qquad \frac{a_{n+1}}{a_n} = \frac{\cancel{a_n} \cdot \frac{c}{n+1}}{\cancel{a_n}} = \frac{c}{n+1} < \text{ počev od nekog } n.$$

Počevši od nekog n, a_n opada $(\forall n \in \mathbb{N})\ a_n > 0 \Rightarrow a_n$ konv. U šta?

$$a_{n+1} = \frac{c}{n+1} \cdot a_n$$
$$L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \underbrace{\left(\frac{c}{n+1}\right)}_{\to 0} \cdot \underbrace{a_n}_{\to L} = 0 \cdot L = 0 \Rightarrow \lim_{n\to\infty} \frac{c^n}{n!} = 0$$

Domaći!

$x_1>0 \quad x_{n+1} = x_n + 1/x_n$ Ispitati konv.

Jagićeva u 3
Nadoknada, četvrtak!

$(1+1/n)^n \nearrow e$

$(1+1/n)^{n+1} \searrow e$

$\ln n \ll n^k \ll e^n \ll n! \ll n^n \quad n\to\infty$

(10) $a_n = 1+1/2+1/3+\dots+1/n$ div.

$a_{2n}-a_n = \frac{1}{n+1}+\frac{1}{n+2}+\dots+\frac{1}{2n} \ge \frac{n}{2n} = 1/2$

$(\forall \varepsilon>0)(\exists n_0\in\mathbb{N})(\forall n\ge n_0)\ |a_n - L|<\varepsilon$

$(\forall m,n\ge n_0)\ |a_n-a_m|<2\varepsilon$

$(\exists\varepsilon>0)(\forall n_0\in\mathbb{N})(\exists n,m\ge n_0)\ |a_n-a_m|\ge 2\varepsilon$ ↯

$\varepsilon = 1/4,\ n=n_0,\ m=2n_0$

(11) $a_n = 1+1/2+\dots+1/n - \ln n$ Dok da konv.

$a_{n+1}-a_n = \frac{1}{n+1} - \ln(n+1)+\ln(n) = \frac{1}{n+1} - \ln(1+1/n) <0$

Pošto važi $e<(1+1/n)^{n+1}\ /\ln$

$1<(n+1)\ln(1+1/n)$

$\frac{1}{n+1}<\ln(1+1/n)$

Probamo da dokažemo $a_n>0$

$(1+1/n)^n<e<(1+1/n)^{n+1}\ /\ln$

$n\ln(1+1/n)<1<(n+1)\ln(1+1/n)$

$\frac{1}{n+1}<\ln(1+1/n)<1/n$

$a_n = 1+1/2+1/3+\dots+1/n-\ln n > \ln 2+\ln(1+1/2)+\dots+\ln\frac{n+1}{n}-\ln n$

$=\ln\left(2\cdot 3/2\cdot 4/3\cdot 5/4\cdots \frac{n+1}{n}\right) - \ln n = \ln\frac{n+1}{n}>0$

(12) $\lim_{n\to\infty} \overbrace{\left(\frac{1}{n+1}+\frac{1}{n+2}+\dots+\frac{1}{2n}\right)}^{b_n} = ?$

$a_{2n}-a_n = \frac{1}{n+1}+\frac{1}{n+3}+\dots+\frac{1}{2n} - \ln(2n)+\ln n$

$b_n = a_{2n} - a_n + \ln 2$

a_n konv. $\to (\exists L \in \mathbb{R})\ L = \lim_{n\to\infty} a_n$

$\Rightarrow \lim_{n\to\infty} a_n = L$

$\lim_{n\to\infty} b_n = L - L + \ln 2 = \ln 2$

Podnizovi i tačke nagomilavanja

$(a_n)_{n\in\mathbb{N}} \rightsquigarrow k: \mathbb{N} \to \mathbb{N}$ rastuća

$(a_{k_n})_{n\in\mathbb{N}}$ podniz niza $(a_n)_{n\in\mathbb{N}}$

X tačka nagomilavanja niza $(a_n)_{n\in\mathbb{N}}$ ako postoji podniz od $(a_n)_{n\in\mathbb{N}}$ koji konv. u X. $\lim_{n\to\infty} a_{k_n} = X$

① $(-1)^n = a_n$ T.N. 1 i -1

$a_n = \sin \frac{n\pi}{2}$ $\quad a_{2n} = 0;\ \lim_{n\to\infty} a_{2n} = 0 \ \Big|\ a_{4n+1} = 1;\ \lim_{n\to\infty} a_{4n+1} = 1 \ \Big|\ a_{4n+3} = -1;\ \lim_{n\to\infty} a_{4n+3} = -1$

$\overline{\lim}\, a_n$: najveća tačka nagomilavanja

$\underline{\lim}\, a_n$: najmanja ——— || ———

① $X_n = 1 + \frac{n^2}{n^2+5} \cos \frac{n\pi}{2}$ $\qquad \cos\frac{n\pi}{2} = \begin{cases} 1, & n = 4k \\ 0, & n = 2k-1 \\ -1, & n = 4k-2 \end{cases}$

$X_{2n-1} = 1 + \frac{(2n-1)^2}{(2n-1)^2+5} \cdot 0 = 1 \longrightarrow 1$

$X_{4n} = 1 + \frac{16n^2}{16n^2+5} \cdot 1 = 2 \longrightarrow 2,\ n\to\infty$

$X_{4n-2} = 1 + \frac{(4n-2)^2}{(4n-2)^2+5}(-1) \longrightarrow 0,\ n\to\infty$

$T(X_n) = \{0, 1, 2\}$

$\overline{\lim}\, X_n = 2 \quad \underline{\lim}\, X_n = 0$

② $a_n = \sqrt[n]{2^n + 5^{(-1)^n \cdot n}}$ $\qquad a_{2n} = \sqrt[2n]{2^{2n} + 5^{2n}}$ $\quad \lim_{n\to\infty} a_{2n} = \max\{2, 5\} = 5$

$a_{2n-1} = \sqrt[2n-1]{2^{2n-1} + (1/5)^{2n-1}}$ $\quad \lim_{n\to\infty} a_{2n-1} = \max\{2, 1/5\} = 2$

$T(X_n) = \{2, 5\}$

Domaći $\quad X_n = (-1)^n (1+ 1/n)^n + \frac{n}{n+1} \sin \frac{2n\pi}{3} + \frac{\ln n}{n!}$

(3) $\lim_{n\to\infty} a_{2n} = a = \lim_{n\to\infty} a_{2n-1} \xRightarrow{?} \lim_{n\to\infty} a_n = a$

$(\forall \varepsilon>0)(\exists n_1 \in \mathbb{N})\ l > n_1 \Rightarrow |a_{2l} - a| < \varepsilon$

$(\exists n_2 \in \mathbb{N})\ m > n_2 \Rightarrow |a_{2m-1} - a| < \varepsilon$

$n_0 = \max\{2n_1, 2n_2\}$

$(\forall \varepsilon>0)(\exists n_0 \in \mathbb{N})\ n \geq n_0$

$n = 2k:\ 2k \geq 2n_1 \Rightarrow |a_{2k} - a| < \varepsilon$

$n = 2k-1:\ 2k-1 \geq 2n_2 - 1 \Rightarrow k \geq n_2 \Rightarrow |a_{2k-1} - a| < \varepsilon$

$\Rightarrow |a_n - a| < \varepsilon$

Poslednji čas

Štolc $\quad (X_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \quad \lim_{n\to\infty} \frac{X_n}{y_n}$

$y_n \nearrow, \lim_{n\to\infty} y_n = +\infty$

Ako postoji $\lim_{n\to\infty} \frac{X_{n+1} - X_n}{y_{n+1} - y_n} = L$, onda je $\lim_{n\to\infty} \frac{X_n}{y_n} = L$

Ako: $\lim \frac{X_n}{y_n} \overset{\text{Štolc}}{=} \lim \frac{X_{n+1} - X_n}{y_{n+1} - y_n} = \ldots =$ ne postoji. Onda preuređujemo

(1) $\lim_{n\to\infty} \frac{1^k + 2^k + \ldots + n^k}{n^{k+1}}$ $\quad 1^k + 2^k + \ldots + n^k \to X_n$, $\quad n^{k+1} \to y_n$

$y_n \nearrow \quad \lim_{n\to\infty} y_n = +\infty$

$$\lim_{n\to\infty} \frac{1^k + 2^k + \ldots + n^k}{n^{k+1}} \overset{\text{Štolc}}{=} \lim_{n\to\infty} \frac{(n+1)^k}{(n+1)^{k+1} - n^{k+1}} = \lim_{n\to\infty} \frac{n^k + \binom{k}{1} n^{k-1} + \ldots + 1}{n^{k+1} + \binom{k+1}{1} n^k + \binom{k+1}{2} n^{k-1} + \ldots + 1 - n^{k+1}}$$

$$= \lim_{n\to\infty} \frac{n^k + \binom{k}{1} n^{k-1} + \ldots + 1}{(k+1) n^k + \binom{k+1}{2} n^{k-1} + \ldots + 1} = \lim \frac{\cancel{n^k}\left(1 + \binom{k}{1} \frac{1}{n} + \binom{k}{2} \frac{1}{n^2} + \ldots + \frac{1}{n^k}\right)}{\cancel{n^k}\left(k+1 + \binom{k+1}{2} \frac{1}{n} + \ldots + \frac{1}{n^k}\right)}$$

$$= \frac{1}{k+1}$$

② $\lim\limits_{n\to\infty}\left(\frac{1^k+2^k+\dots+n^k}{n^k}-\frac{n}{k+1}\right)=\lim\limits_{n\to\infty}\frac{(k+1)(1^k+2^k+\dots+n^k)-n^{k+1}}{(k+1)n^k}=$

$\left[(k+1)n^k\nearrow \qquad \lim (k+1)n^k=+\infty\right]$

Štolc
$$=\lim_{n\to\infty}\frac{(k+1)(n+1)^k-(n+1)^{k+1}+n^{k+1}}{(k+1)((n+1)^k-n^k)}=\lim_{n\to\infty}\frac{(k+1)\left(n^k+\binom{k}{1}n^{k-1}+\dots+1\right)-n^{k+1}-\binom{k+1}{1}n^k-\dots-1+n^{k+1}}{(k+1)\left(n^k+\binom{k}{1}n^{k-1}+\binom{k}{2}n^{k-2}+\dots+1-n^k\right)}$$

$$=\lim_{n\to\infty}\frac{\cancel{(k+1)n^k}+(k+1)\binom{k}{1}n^{k-1}+\dots+(k+1)-\cancel{(k+1)n^k}-\binom{k+1}{2}n^{k-1}-\dots-\binom{k+1}{k}n-1}{(k+1)n^{k-1}\left(k+\binom{k}{2}\frac{1}{n}+\dots+\frac{1}{n^{k-1}}\right)}$$

$$=\lim_{n\to\infty}\frac{\cancel{n^{k-1}}\left(\left((k+1)\binom{k}{1}-\binom{k+1}{2}\right)+\left((k+1)\binom{k}{2}-\binom{k+1}{3}\right)\frac{1}{n}+\dots+((k+1)-1)\frac{1}{n^{k-1}}\right)}{(k+1)\cancel{n^{k-1}}\left(k+\binom{k}{2}\frac{1}{n}+\dots+\frac{1}{n^{k-1}}\right)}$$

$$=\frac{(k+1)k-\frac{(k+1)k}{2}}{(k+1)k}=\boxed{1/2}$$

Tvrdenje: $\lim\limits_{n\to\infty}a_n=a \Rightarrow \lim\limits_{n\to\infty}\frac{a_1+\dots+a_n}{n}=a$

$a_n>0$ $\quad \lim\limits_{n\to\infty}a_n=a \Rightarrow \lim\limits_{n\to\infty}\sqrt[n]{a_1\cdot\ldots\cdot a_n}=a$

(K)

b_{n+1} $\quad b_1=a_1$

$a_n>0$

$\exists\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n}=L \Rightarrow \lim\limits_{n\to\infty}\sqrt[n]{a_n}=L$

$\Rightarrow \lim\limits_{n\to\infty}\sqrt[n]{b_1 b_2\cdots b_n}=L$

$\Rightarrow \lim\limits_{n\to\infty}\sqrt[n]{a_1\cdot\frac{a_2}{a_1}\cdot\frac{a_3}{a_2}\cdot\ldots\cdot\frac{a_n}{a_{n-1}}}=\lim\limits_{n\to\infty}\sqrt[n]{a_n}$

$b_n=\frac{a_n}{a_{n-1}} \qquad b_1=a_1, \quad b_2=\frac{a_2}{a_1}$

③ a_n

$\lim\limits_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\lim\limits_{n\to\infty}\sqrt[n]{\frac{n!}{n^n}}$

$\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim\limits_{n\to\infty}\frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}}=\lim\limits_{n\to\infty}\frac{(n+1)n^n}{(n+1)^{n+1}}=\lim\limits_{n\to\infty}\left(\frac{n}{n+1}\right)^n=$

$=\lim\limits_{n\to\infty}\frac{1}{\left(\frac{n+1}{n}\right)^n}=\lim\limits_{n\to\infty}\frac{1}{(1+1/n)^n}=1/e$

4) $a_0 > 0$, $a_{n+1} = a_n + \frac{1}{3 \cdot a_n^2}$, $\forall n \geq 0$

a) $\lim_{n\to\infty} a_n$

$a_{n+1} - a_n = \frac{1}{3a_n^2} > 0 \Rightarrow a_n \nearrow$ rastući niz

Ako bi $(a_n)_{n\in\mathbb{N}}$ bio ograničen $\Rightarrow$ konvergentan

$\Rightarrow \lim_{n\to\infty}(a_{n+1} - a_n) = 0 \Rightarrow \lim_{n\to\infty} \frac{1}{3a_n^2} = 0 \Rightarrow \lim_{n\to\infty} a_n = +\infty$ $\Big\}$ $\not\!\!\!\downarrow$

$\Rightarrow (a_n)_{n\in\mathbb{N}}$ nije ograničen $\Rightarrow \lim_{n\to\infty} a_n = +\infty$

b) $\lim_{n\to\infty} \frac{a_n^3}{n} \overset{\text{Štolc}}{=} \lim_{n\to\infty} \frac{a_{n+1}^3 - a_n^3}{1} = \lim_{n\to\infty}\left(\left(a_n + \frac{1}{3a_n^2}\right)^3 - a_n^3\right)$

$= \lim_{n\to\infty}\left(\cancel{a_n^3} + \cancel{3}\cancel{a_n^2} \cdot \frac{1}{\cancel{3}\cancel{a_n^2}} + \cancel{3a_n} \frac{1}{\cancel{9}_3 a_n^{\cancel{4}\,3}} + \frac{1}{27a_n^6} - \cancel{a_n^3}\right)$

$= \lim_{n\to\infty}\left(1 + \frac{1}{3} \cdot \frac{1}{a_n^3} + \frac{1}{27} \cdot \frac{1}{a_n^6}\right) = 1$

5) $x_0 > 0$ $\quad x_{n+1} = \frac{x_n}{1 + x_n^2}$, $n \geq 0$

a) $\lim_{n\to\infty} x_n = ?$

$x_{n+1} \leq x_n \quad \frac{x_n}{1 + x_n^2} \leq x_n \quad \frac{1}{1+x_n^2} \leq 1 \quad 1 \leq 1 + x_n^2$

$0 \leq x_n^2$ ✓

$\left.\begin{array}{l} \Rightarrow x_n \downarrow \\ x_n \geq 0 \end{array}\right\} \Rightarrow (x_n)_{n\in\mathbb{N}}$ konvergira

$L = \lim x_n = \lim x_{n+1}$

$= \lim \frac{x_n}{1 + x_n^2} = \frac{L}{1 + L^2}$ $\qquad L = \frac{L}{1+L^2}$ $\begin{array}{l} L = 0 \\ L \neq 0 \quad \cancel{1 = \frac{1}{1+L^2} \Rightarrow L = 0} \end{array}$

b) $\lim_{n\to\infty} n \cdot x_n^2 = ?$

$\lim_{n\to\infty} \frac{1}{n \cdot x_n^2} = \lim_{n\to\infty} \frac{\frac{1}{x_n^2}}{n} \overset{\text{štolc}}{=} \lim_{n\to\infty} \frac{\frac{1}{x_{n+1}^2} - \frac{1}{x_n^2}}{n+1-n}$

$= \lim_{n\to\infty}\left(\left(\frac{1 + x_n^2}{x_n}\right)^2 - \frac{1}{x_n^2}\right) = \lim_{n\to\infty} \frac{1 + x_n^4 + 2x_n^2 - 1}{x_n^2} = \lim_{n\to\infty}(2 + \cancel{x_n^2}) = 2$

$= \lim_{n\to\infty} n \cdot x_n^2 = \frac{1}{\lim_{n\to\infty} \frac{1}{n x_n^2}} = \boxed{\frac{1}{2}}$

⑥ $$\lim_{n\to\infty}\left(\frac{n^2+2n+1}{n^2-n+1}\right)^{\frac{n^2+1}{n+1}} = \lim_{n\to\infty}\left(1+\frac{3n+1}{n^2-n+1}\right)^{\frac{n^2+1}{n+1}}$$

$$\lim_{n\to\infty}\left(1+\frac{1}{\frac{n^2-n+1}{3n+1}}\right)\wedge\left(\frac{(n^2-n+1)(3n+1)(n^2+1)}{(n^2-n+1)(3n+1)(n+1)}\right) \lim_{n\to\infty} e \wedge \left(\frac{(3\cancel{n}+1)(n^{\cancel{2}}+1)}{(\cancel{n}+1)(\cancel{n^2}-n+1)}\right) = \boxed{e^3}$$

⑦ $$X_n = \underbrace{(-1)^n}_{\substack{n \text{ parno} \\ n \text{ neparno}}}\underbrace{(1+1/n)^n}_{e} + \underbrace{\frac{n}{n+1}}_{1}\cdot\underbrace{\sin\frac{2n\pi}{3}}_{\substack{n\equiv 1 \,(\mathrm{mod}\ 3)\\ n\equiv 2\,(\mathrm{mod}\ 3)\\ n\equiv 0\,(\mathrm{mod}\ 3)}} + \underbrace{\frac{\ln n}{n!}}_{0}$$

$$\sin\frac{2n\pi}{3} = \begin{cases} 0, & n\equiv_3 0 \\ \frac{\sqrt3}{2}, & n\equiv_3 1 \\ -\frac{\sqrt3}{2}, & n\equiv_3 2\end{cases}$$

$n=6k$; $\lim_{x\to\infty} X_{6k} = 1\cdot e + 1\cdot 0 + 0 = e$

$n=6k+1$; $\lim_{x\to\infty} X_{6k+1} = -e + 1\cdot\frac{\sqrt3}{2} + 0 = -e + \frac{\sqrt3}{2}$

$n=6k+2$; $\lim_{x\to\infty} X_{6k+2} = e + 1\cdot\left(-\frac{\sqrt3}{2}\right) + 0 = e - \frac{\sqrt3}{2}$

$n=6k+3$; $\lim_{x\to\infty} X_{6k+3} = -e + 1\cdot 0 + 0 = -e$

$n=6k+4$; $\lim_{x\to\infty} X_{6k+4} = e + 1\cdot\left(\frac{\sqrt3}{2}\right) + 0 = e + \frac{\sqrt3}{2}$

$n=6k+5$; $\lim_{x\to\infty} X_{6k+5} = -e + 1\cdot\left(-\frac{\sqrt3}{2}\right) + 0 = -e - \frac{\sqrt3}{2}$

$\ulcorner a_n = \sqrt{2+\sqrt{2+\dots+\sqrt2}}$ $\quad a_{n+1} = \sqrt{2+a_n}$ $\quad a_1 = \sqrt2$ $\quad a_{n+1} \geq a_n \iff a_n \leq \sqrt2$ w

$a_n \nearrow$

konvergira

$\lim_{n\to\infty} a_n = 2$

⑧ $$\lim_{n\to\infty}\left(\frac{a_n}{2}\right)^n$$

Monotonost: $a_{n+1} \geq a_n$ $\qquad \left(\frac{a_n}{2}\right)^n \leq \left(\frac{a_{n+1}}{2}\right)^{n+1}$

$\iff \frac{a_n^{\,n}}{2^n} \leq \frac{(a_{n+1})^{n+1}}{2^{n+1}}$

$\iff 2a_n^{\,n} \leq (a_{n+1})^{n+1}$

$2a_n^{\,n} \leq \sqrt{2+a_n}^{\,n+1} \quad /^2$

$4a_n^{\,2n} \leq (2+a_n)^{n+1}$

$$4a_n^{\,2n} \leq \underbrace{2^{n+1} + \binom{n+1}{1}2^n a_n + \dots + \binom{n+1}{n}2a_n^{\,n} + a_n^{\,n+1}}_{2^{n+1}\text{ sabiraka}}$$

AG:

$\frac{a+b}{2} \geq \sqrt{ab}$

$(2+a_n)^{n+1} \geq 2^{n+1}a_n^{\,n+1} \geq 2^2\cdot a_n^{\,n-1}\cdot a_n^{\,n+1} = 2^2\cdot a^{2n}$

AG sredina

$2^{n-1} \geq a_n^{\,n-1}$

$\ulcorner 2+a_n \geq 2\sqrt{2a_n}$

$\sqrt{2+a_n}^{\,n+1} \geq (2\sqrt{2a_n})^{\frac{n+1}{2}} = 2^{\frac{n+1}{2}}\cdot 2^{\frac{n+1}{4}}\cdot a_n^{\frac{n+1}{4}} = 2^{3/4(n+1)}\cdot a_n^{\frac{n+1}{4}}$

$= 2^{3/4(n+1)}\cdot a_n^{\frac{n+1}{4}} \geq 2a_n^{\,n} \Rightarrow 2 \geq a_n$

$$\Rightarrow \left(\frac{a_n}{2}\right)^n \nearrow$$
$$\left.\begin{array}{l} \\ \frac{a_n}{2} \leq 1 \Rightarrow \left(\frac{a_n}{2}\right)^n \leq 1 \end{array}\right\} \left(\frac{a_n}{n}\right)^n \text{ konvergira}$$

$$L = \lim_{n\to\infty} \frac{a_n^n}{2^n} = \lim_{n\to\infty} \frac{a_{n+1}^{n+1}}{2^{n+1}} = \lim_{n\to\infty} \frac{\sqrt{2+a_n}^{\,n+1}}{2^{n+1}} \geq \lim \frac{\sqrt{2\sqrt{2a_n}}^{\,n+1}}{2^{n+1}}$$

$$= \lim_{n\to\infty} \frac{2^{\frac{n+1}{n}} 2^{\frac{n+1}{4}} \cdot a_n^{\frac{n+1}{4}}}{2^{n+1}} = \lim_{n\to\infty} \frac{a_n^{\frac{n+1}{4}}}{2^{\frac{n+1}{4}}} = \lim \left(\left(\frac{a_n}{2}\right)^n\right)^{1/4} \cdot \left(\frac{a_n}{2}\right)^{1/4}$$

$\left(\frac{a_n}{2}\right)^n \to L$, $\left(\frac{a_n}{2}\right)^{1/4} \to 1$

$$= \sqrt[4]{L}$$

$$\left.\begin{array}{l} L \geq \sqrt[4]{L} \Rightarrow L^{3/4} \geq 1 \\ \left(\frac{a_n}{2}\right)^n \leq 1 \Rightarrow L \leq 1 \end{array}\right\} L = 1$$

① $f(x) = -\frac{1}{|x|} + \operatorname{arctg} \frac{2x}{x^2-1}$

② $f(x) = \ln(3e^x - 3 + e^{-x})$

① ① Domen: $\frac{1}{|x|} \Rightarrow x \neq 0$ $\quad \frac{2x}{x^2-1} \Rightarrow (x-1)(x+1) \neq 0, x \neq \{-1, 1\}$ $D_f = \mathbb{R}\setminus\{0,1,-1\}$

② Nule i znak; kasnije

③ Nije periodična

Parnost $\quad f(-x) = -\frac{1}{|x|} + \operatorname{arctg} \frac{-2x}{x^2-1} = -\frac{1}{|x|} - \operatorname{arctg} \frac{2x}{x^2-1}$ $\quad$ Ni parna, ni neparna

$f(2) = -1/2 + \operatorname{arctg} 4/3 \qquad f(-2) = -1/2 - \operatorname{arctg} 4/3$

$f(2) = f(-2) \Leftrightarrow \operatorname{arctg} 4/3 = -\operatorname{arctg} 4/3$ ↯

$f(-2) = -f(2) \Leftrightarrow -1/2 = 1/2$ ↯

④ Asimptote

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} \left(\frac{1}{x} + \operatorname{arctg} \frac{2x}{x^2-1}\right) = -1 - \pi/2$$

$\frac{1}{x} \to -1$, $2x \to -2$, $x^2-1 \to 0^+$, $\frac{2x}{x^2-1} \to -\infty$

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} \left(1/x + \operatorname{arctg} \frac{2x}{x^2-1}\right) = -1 + \pi/2$$

$1/x \to -1$, $2x \to -2$, $x^2-1 \to 0^-$, $\frac{2x}{x^2-1} \to \infty$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \left(\frac{1}{x} + \operatorname{arctg} \frac{2x}{x^2-1}\right) = -\infty$$

$\frac{2x}{x^2-1} \to 0$, $\operatorname{arctg} \frac{2x}{x^2-1} \to 0$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \left(-1/x + \operatorname{arctg} \frac{2x}{x^2-1}\right) = -\infty$$

$\operatorname{arctg} \frac{2x}{x^2-1} \to 0$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}\left(-\tfrac{1}{x} + \operatorname{arctg}\frac{2x}{x^2-1}\right) = -1 - \pi/2$$

0^-, $\to -\infty$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}\left(-\tfrac{1}{x} + \operatorname{arctg}\frac{2x}{x^2-1}\right) = -1 + \pi/2$$

$\to +\infty$

Kose / Horizontalne

0, 0

$$\lim_{x\to +\infty}\left(-\tfrac{1}{x} + \operatorname{arctg}\frac{2x}{x^2-1}\right) = 0 \qquad x\to+\infty \quad y=0$$

$$\lim_{x\to -\infty}\left(\frac{1}{x} + \operatorname{arctg}\frac{2x}{x^2-1}\right) = 0 \qquad x\to-\infty \quad y=0$$

5 Monotonost

$x>0$ $\quad f(x) = -\frac{1}{x} + \operatorname{arctg}\frac{2x}{x^2-1}$

$$f'(x) = \frac{1}{x^2} + \frac{1}{1+\left(\frac{2x}{x^2-1}\right)^2}\cdot\frac{2(x^2-1)-4x^2}{(x^2-1)^2} = \frac{1}{x^2} + \frac{-2(1+x^2)}{(x^2-1)^2+(2x)^2}$$

$$= \frac{1}{x^2} + \frac{-2(x^2+1)}{x^4+2x+1} = \frac{1}{x^2} - \frac{2(x^2+1)}{(x^2+1)^2} = \frac{x^2+1-2x^2}{x^2(x^2+1)} = \frac{-x^2+1}{x^2(x^2+1)} \quad >0$$

$f(x)>0$ za $x<1$, $x\in(0,1)$ $\quad f'(x)<0$ za $x\in(1,+\infty)$

$x<0$

$$f'(x) = -\tfrac{1}{x^2} - \frac{2}{x^2+1} < 0 \qquad f \downarrow \text{ na } (-\infty,-1)\cup(-1,0)$$

6. Konkavnost

I $x>0$

$$f'(x) = \frac{1}{x^2} - \frac{2}{x^2+1} \qquad f''(x) = -\frac{2}{x^3} + \frac{2\cdot 2x}{(x^2+1)^2} = \frac{-2(x^2+1)^2 + 4x^4}{x^3(x^2+1)^2}$$

$$= \underbrace{\frac{-2x^4-4x^2-2+4x^4}{x^3(x^2+1)^2}}_{*} \Rightarrow \operatorname{sgn}(*) = \operatorname{sgn}(+2x^4-4x^2-2)$$

$$x_{1/2} = \frac{4\pm\sqrt{16+16}}{2} = \frac{4\pm4\sqrt{2}}{2} = 2\pm2\sqrt{2}$$

$$\operatorname{sgn} 2\left(x^2-(1-\sqrt{2})\right)\left(x^2-(1+\sqrt{2})\right) = \operatorname{sgn}\underbrace{(x^2+\sqrt{2}-1)}_{>0}(x^2-\sqrt{2}-1)$$

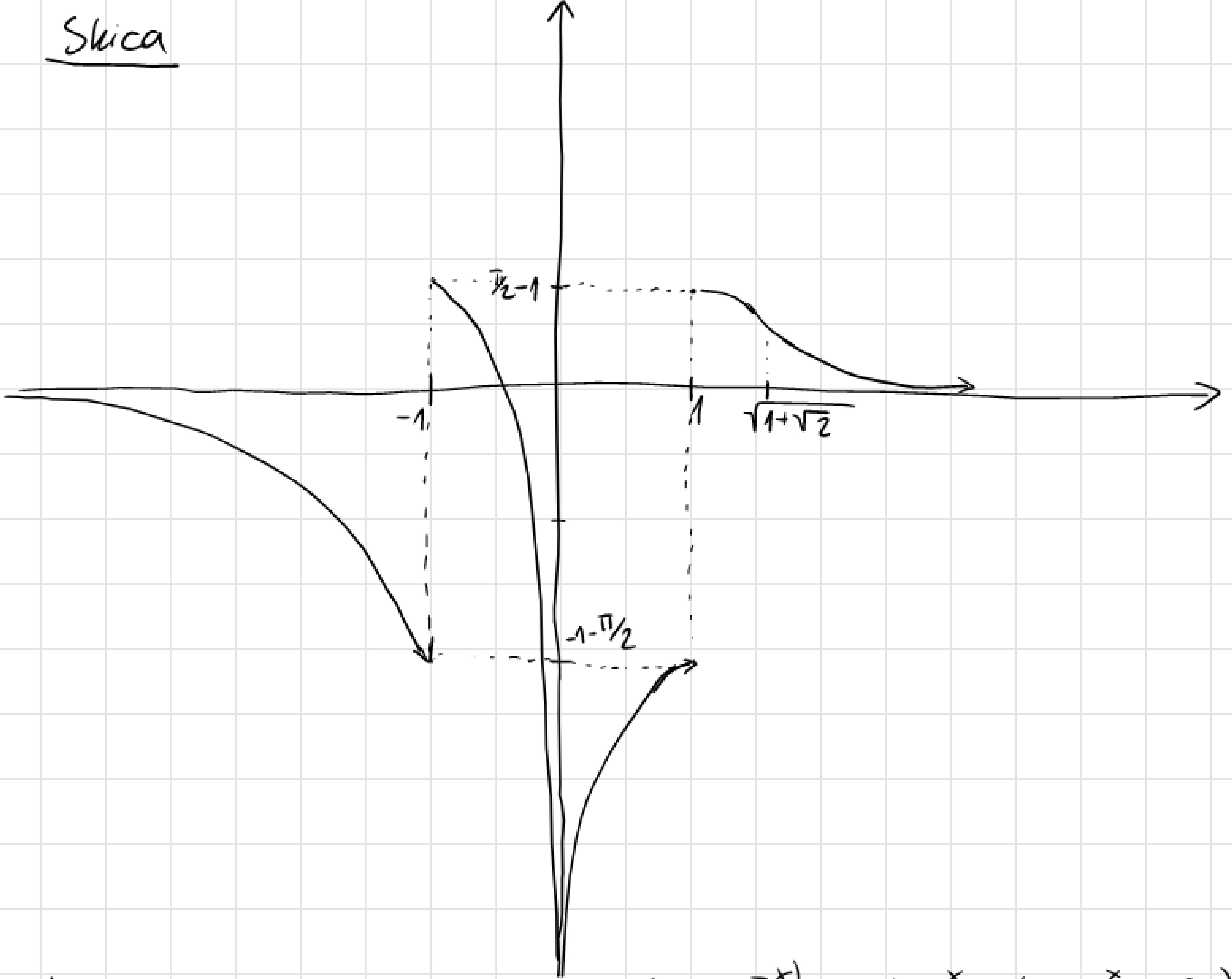
$x^2-\sqrt{2}-1$ $\quad f''(x)>0$ za $x^2>1+\sqrt{2}$ tj $x>\sqrt{1+\sqrt{2}}$

$\qquad\qquad <0 \quad x^2<1+\sqrt{2}$ tj $x<\sqrt{1+\sqrt{2}}$

II $x<0$ $\quad f'(x) = -\frac{1}{x^2} - \frac{2}{x^2+1} \qquad f''(x) = \frac{2(x^2+1)^2+4x^4}{x^3(x^2+1)^2} < 0$

>0, $0>$, >0

Skica



Uputstvo za 2)
Kosa asimptota

$$\lim \frac{\ln(3e^{x}-3+e^{-x})}{x} = \frac{\ln e^{x} + \ln(3-3e^{-x}+e^{-2x})}{x}$$

$$= 1 + \frac{\ln(3-3/e^{x}+e/e^{2x})}{x} =$$

$$= 1 + \frac{\ln 3}{x} = 1$$

$$\overset{LOP.}{=} \lim \frac{1}{3e^{x}-3+e^{-x}}\cdot(3e^{x}-e^{-x}) = \lim \frac{e^{x}(3-e^{-2x})}{e^{x}(3-3e^{-x}+e^{-2x})} = 3/3 = 1$$