Čas 1

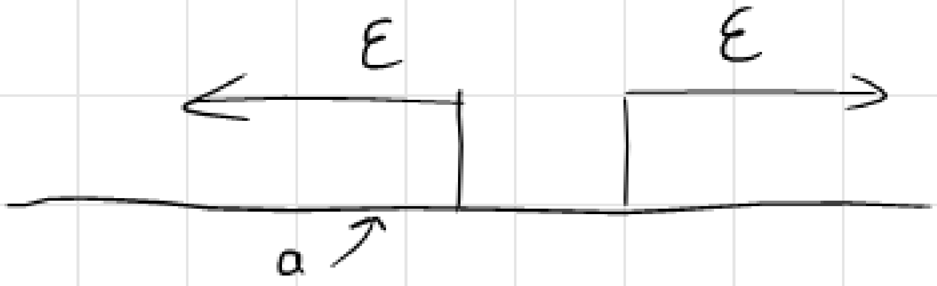
Supremum i infimum

$\sup A = S \quad A \subseteq K \quad , S \geq A$

$(\forall x \in K)\, x \geq A \Rightarrow x \geq S$

$A \subseteq \mathbb{R} \quad \sup A = S \quad S \geq A$

$(\forall \varepsilon > 0)(\exists a \in A)\, a > S - \varepsilon$



$\inf A = p \quad p \leq A \quad (\forall \varepsilon > 0)(\exists a \in A)\, a < p + \varepsilon$

1. $A = \{ \frac{1}{n} \mid n \in \mathbb{N} \}$

$\frac{1}{n} \searrow \quad \max A = 1 \quad \frac{1}{n} \leq 1$

$\inf A = 0$

$\frac{1}{n} \geq 0 \quad (\forall n \in \mathbb{N})\ \frac{1}{n} \geq 0 \Rightarrow A \geq 0 \quad \text{I}$

uzmimo proizvoljno $\varepsilon > 0$ (bez gubitka opštosti)

? $(\exists a \in A)\, a < \varepsilon$ → ceo deo

$\frac{1}{n} < \varepsilon \Leftrightarrow n > \frac{1}{\varepsilon} \quad n_0 = \lfloor \frac{1}{\varepsilon} \rfloor + 1 \quad \underline{n_0 \in \mathbb{N}}$

$\frac{1}{n_0} \in A \quad n_0 > \frac{1}{\varepsilon} \quad \lfloor \frac{1}{\varepsilon} \rfloor + 1 \Rightarrow \frac{1}{n_0} < 0 + \varepsilon$

Dakle $(\forall \varepsilon > 0)(\exists a \in A)\, a < \varepsilon + 0 \quad \text{II} \quad \text{I} \wedge \text{II} \Rightarrow 0 = \inf A$

2. $A=\left\{\frac{9n-4}{2n+2} \,\middle|\, n\in N\right\}$

$$\frac{9n-4}{2n+2}=\frac{9(n+1)-9-4}{2(n+1)}=\frac{9}{2}-\frac{13}{2(n+1)}\nearrow$$

$a_{n+1}\square a_n$

$\frac{9(n+1)-4}{2(n+1)+2}\square\frac{9n-4}{2n+2}$ $\qquad$ $\frac{9n+5}{2n+4}\square\frac{9n-4}{2n+2}$

$(9n+5)(n+1)\square(9n-4)(n+2)$

$9n^2+9n+5n+5\square 9n^2+18n-4n-8$

$\underline{9n^2}+\underline{14n}+5\square \underline{9n^2}+\underline{14n}-8$

$5\boxed{>}-8 \Rightarrow a_{n+1}>a_n$

$a_1=\frac{9-4}{4}=\frac{5}{4}$ $\quad a_1<a_2<\dots<a_n \Rightarrow \frac{5}{4}=\min A$

$\frac{5}{4}\le A$ $\quad 5/4\le\frac{9n-4}{2n+2}$ za svako $n\in N$

$5(2n+2)\le 4(9n-4)$ $\qquad 10n+10\le 36n-16$

$26\le 26n$

$1\le n$

$\left.\begin{array}{l}\frac{5}{4}\in A\\ \frac{5}{4}\le A\end{array}\right\}$ min

$\sup A=9/2$

I: $9/2\ge A \Leftrightarrow (\forall a\in N)\ \frac{9}{2}\ge\frac{9n-4}{2n+2}\Leftrightarrow 18n+18\ge 18n-8$

$18\ge -8$ W $\quad 9/2\ge A$

II: $\varepsilon>0$ proizvoljno

$(\exists a\in A)\ a>9/2-\varepsilon$ $\qquad \frac{9n-4}{2n+2}>\frac{9}{2}-\varepsilon\Leftrightarrow \varepsilon>\frac{9}{2}-\frac{9n-4}{2n+2}$

$\Leftrightarrow \varepsilon>\frac{9n+9-9n+4}{2(n+1)}\Leftrightarrow \varepsilon>\frac{13}{2(n+1)}$ $\qquad n_0=\left\lfloor\frac{13}{2\varepsilon}\right\rfloor$

$n_0+1>\frac{13}{2\varepsilon}$

$\Leftrightarrow n+1>\frac{13}{2\varepsilon}$

Nastavak

$n_0 \in N$ $\quad \frac{9n_0-4}{2n_0+2} \in A \quad \frac{9n_0-4}{2n_0+2} > \frac{9}{2} - \varepsilon$ $\quad I \wedge II \Rightarrow \sup A = 9/2$

Dakle $(\forall \varepsilon > 0)(\exists a \in A)\, a > \frac{9}{2} - \varepsilon$

3. $A = \{(2\sin(3x+\pi)) \mid x \in [-\frac{\pi}{12}, \frac{\pi}{6})\}$

$x \in [-\frac{\pi}{12}, \frac{\pi}{6}) \quad 3x \in [-\frac{\pi}{4}, \frac{\pi}{2}) \quad 3x+\pi \in [\frac{3\pi}{4}, \frac{3\pi}{2})$

$A = \{2\sin y \mid y \in [\frac{3\pi}{4}, \frac{3\pi}{2})\}$

$\underline{A = (-2, \sqrt{2}]}$

$\max A = \sqrt{2} \quad \underset{\in A}{2\sin\frac{3\pi}{4} = \sqrt{2}} \quad 2\sin y \le \sqrt{2} \quad y \in [\frac{3\pi}{4}, \frac{3\pi}{2})$

$y \in [\pi, \frac{3\pi}{2}) \Rightarrow \sin y \le 0 < \frac{\sqrt{2}}{2}$

$y \in [\frac{3\pi}{4}, \pi) \Rightarrow \sin y = \sin(\underbrace{\pi - y}_{\in (0, \frac{\pi}{4}]}) \le \frac{\sqrt{2}}{2}$

$\Rightarrow A \le \sqrt{2} \wedge \sqrt{2} \in A \quad \max A = \sqrt{2}$

$? \inf A = -2$

$I \quad \sin y \ge -2$

$2\sin y \ge -2 \leftarrow$ za bilo koje y

$\Rightarrow A \ge -2$

$II \quad 1 > \varepsilon > 0$ proizvoljno

$?(\exists a \in A)\, a < -2 + \varepsilon$

$2\sin y < -2 + \varepsilon$

$\sin y < -1 + \frac{\varepsilon}{2}$

$y \in [\frac{3\pi}{4}, \frac{3\pi}{2})$

$-1 + \frac{\varepsilon}{4} < -1 + \frac{\varepsilon}{2}$

$\pi - \arcsin(-1 + \frac{\varepsilon}{4}) = y_0$

Nastavak

$\sin y_0 = \sin(\pi - \arcsin(-1+\varepsilon/4)) = \sin(\arcsin(-1+\frac{\varepsilon}{4})) = -1+\frac{\varepsilon}{4}$

$-1+\frac{\varepsilon}{4} < -1+\frac{\varepsilon}{2}$ $\quad \arcsin(-1+\frac{\varepsilon}{4}) \in (-\frac{\pi}{2}, 0)$

$\pi - \arcsin(-1+\frac{\varepsilon}{4}) \in (-\frac{\pi}{2}, 0)$

$\pi - \mathrm{arsin}(-1+\varepsilon/4) \in (\pi, \frac{3\pi}{2})$ $\quad y_0 \in (\pi, \frac{3\pi}{2})$

$\iff y_0 \in [\frac{3\pi}{4}, \frac{3\pi}{2})$

$\implies 2\sin y_0 \in A$

Dakle $(\forall \varepsilon > 0)(\exists \underset{\hookrightarrow 2\sin y_0}{a} \in A)\ a < -2+\varepsilon$

$I \land II \implies \inf A = -2$

4. $A = \{\frac{m}{m+n} \mid m, n \in \mathbb{N}\}$

fiksiramo m: $\frac{m}{m+n} \downarrow$ kako n raste

fiksiramo n $\frac{m}{m+n} \to 1 - \frac{n}{m+n} \nearrow$ kako m raste

$\inf A = 0$

I $A > 0$ $\forall m, n \in \mathbb{N}$ $\frac{m}{m+n} > 0$

II proizvoljno $\varepsilon > 0$ $m = 1$ $(\exists a \in A)\ a < \varepsilon$ $\quad \frac{1}{1+n} < \varepsilon$

$\iff n+1 > \frac{1}{\varepsilon}$ $\quad n_0 = \lfloor \frac{1}{\varepsilon} \rfloor$ $\quad n_0 + 1 > \frac{1}{\varepsilon}$ $\quad \frac{1}{n+n_0} \in A$ $\quad \frac{1}{1+n_0} < \varepsilon$

Dakle $(\forall \varepsilon > 0)(\exists \underset{\uparrow \frac{1}{1+n_0}}{a} \in A)\ a < 0 + \varepsilon$

$I \land II \implies \inf A = 0$

$\sup A = 1$

I $m < m+n$ $\quad \frac{m}{m+n} < 1$ $\forall m, n \in \mathbb{N}$ $\implies A < 1$ w

II $\varepsilon > 0$ proizvoljno

$(\exists a \in A)\ a > 1 - \varepsilon$ $\quad \frac{m}{m+1} > 1 - \varepsilon \iff \varepsilon > 1 - \frac{m}{m+1} = \frac{1}{m+1}$

$\iff m+1 > \frac{1}{\varepsilon}$ $\quad m_0 = \lfloor \frac{1}{\varepsilon} \rfloor \implies m_0 + 1 > \frac{1}{\varepsilon}$

$\implies \frac{m_0}{m_0+1} \in A > 1 - \varepsilon$ $\quad$ Dakle $(\forall \varepsilon > 0)(\exists \underset{\uparrow \frac{m_0}{m_0+1}}{a} \in A)\ a > 1 - \varepsilon$

$I \land II \implies \sup A = 1$

Domaći

5. $A=\left\{\frac{m}{n} \mid m,n\in\mathbb{N} \quad m<3n\right\}$

6. $A=\left\{\frac{m}{n}+\frac{4n}{m} \mid m,n\in\mathbb{N} \quad m<10n\right\}$

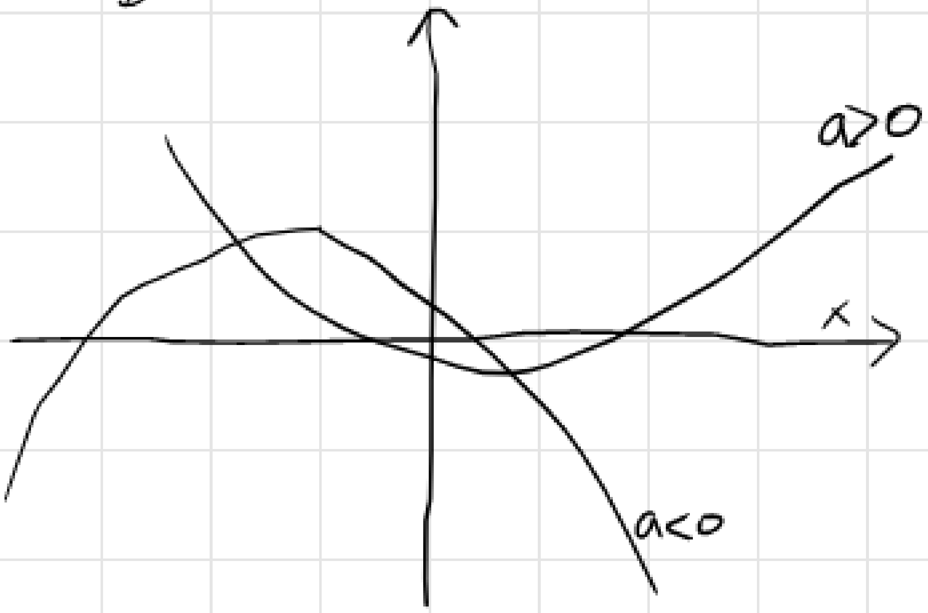
7. $A=\left\{\frac{n+1}{2n}\cdot\frac{2+3m}{m+1} \mid m,n\in\mathbb{N}\right\}$

8. $A=\left\{\sin\left(\frac{n\pi}{2}\right)\frac{n^2-9}{5n^2+2} \mid n\in\mathbb{N}\right\}$

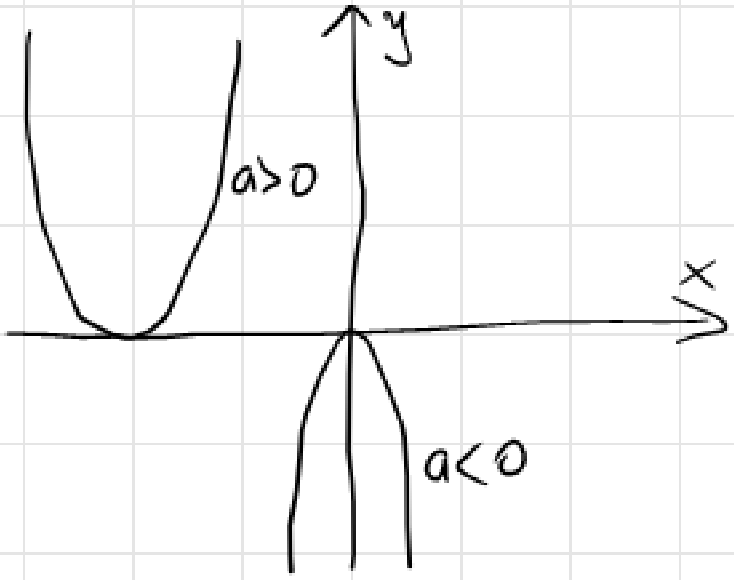
Crtanje grafika funkcije

1) $f(x)=ax^2+bx+c \quad D=b^2-4ac$

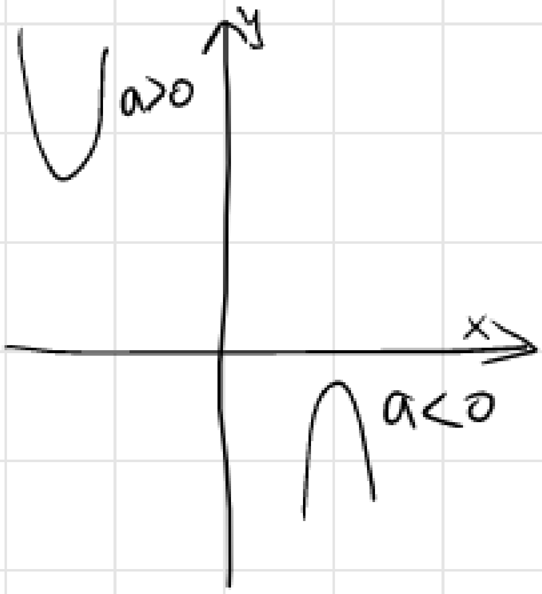
I $D>0$



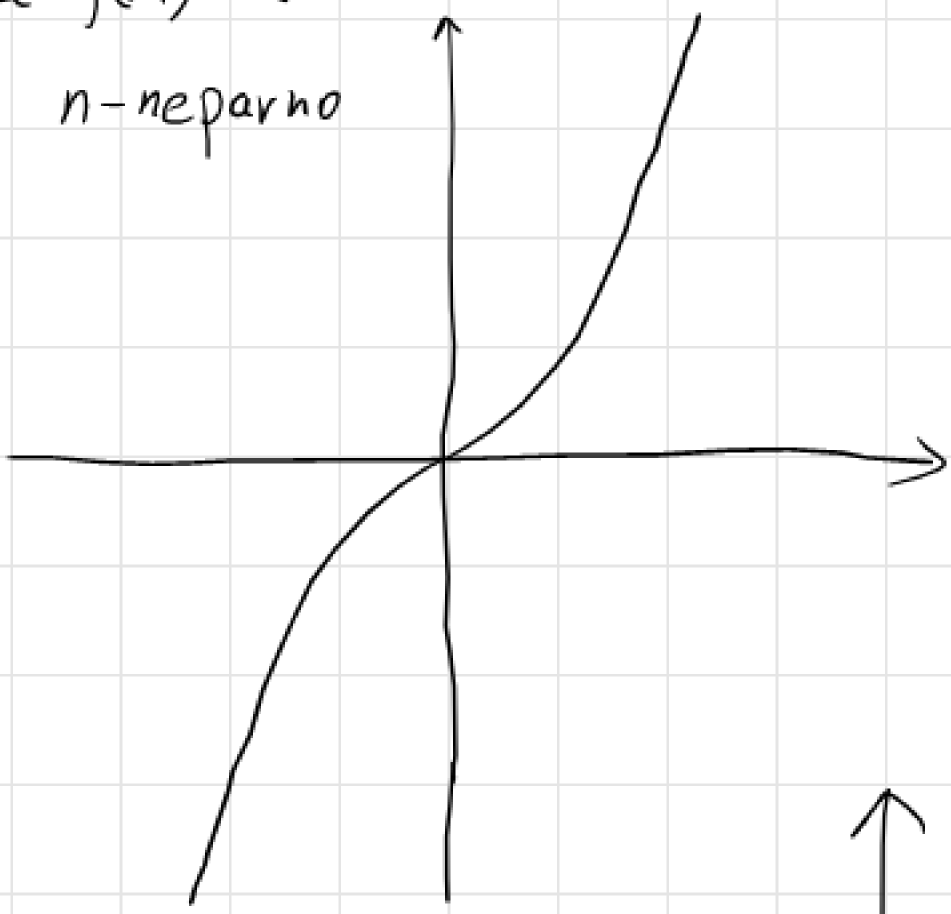
$D=0$

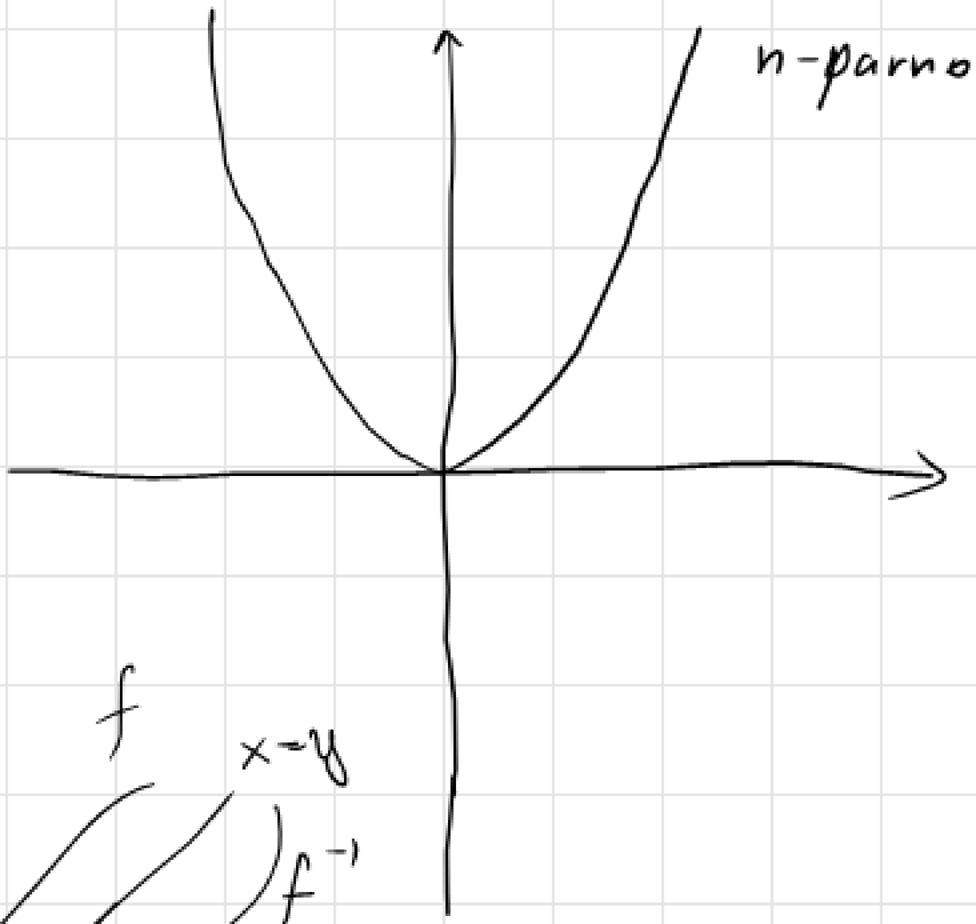


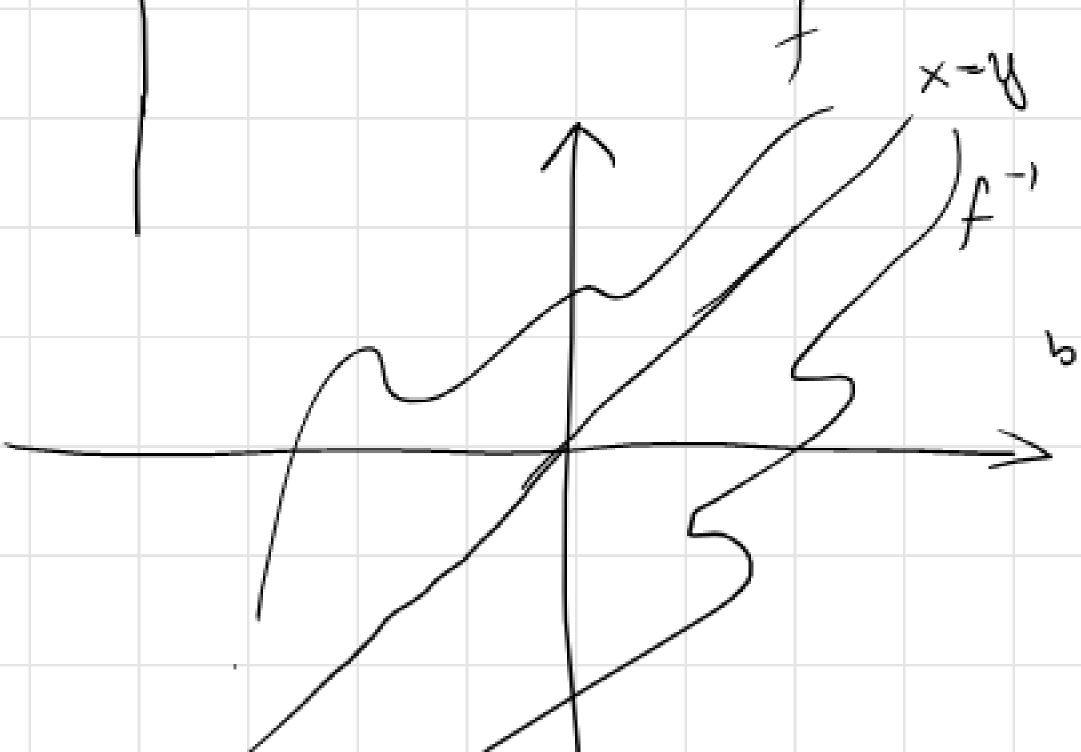
$D<0$



2. $f(x)=x^n$

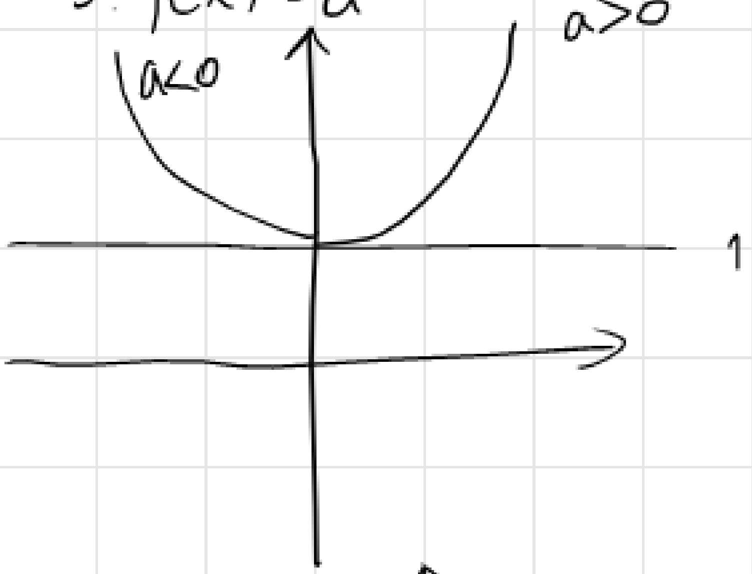






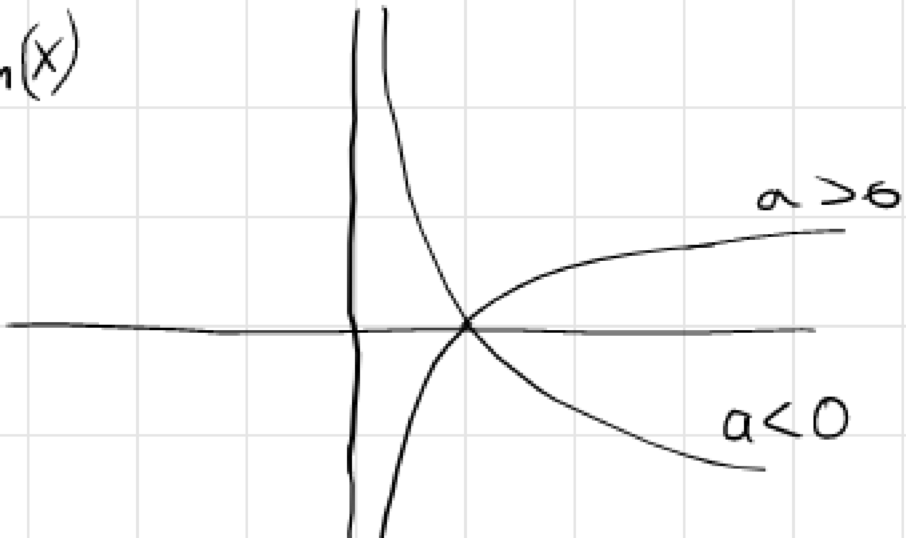
bijektivna f-ja
može imati inverz

3. $f(x) = a^x$

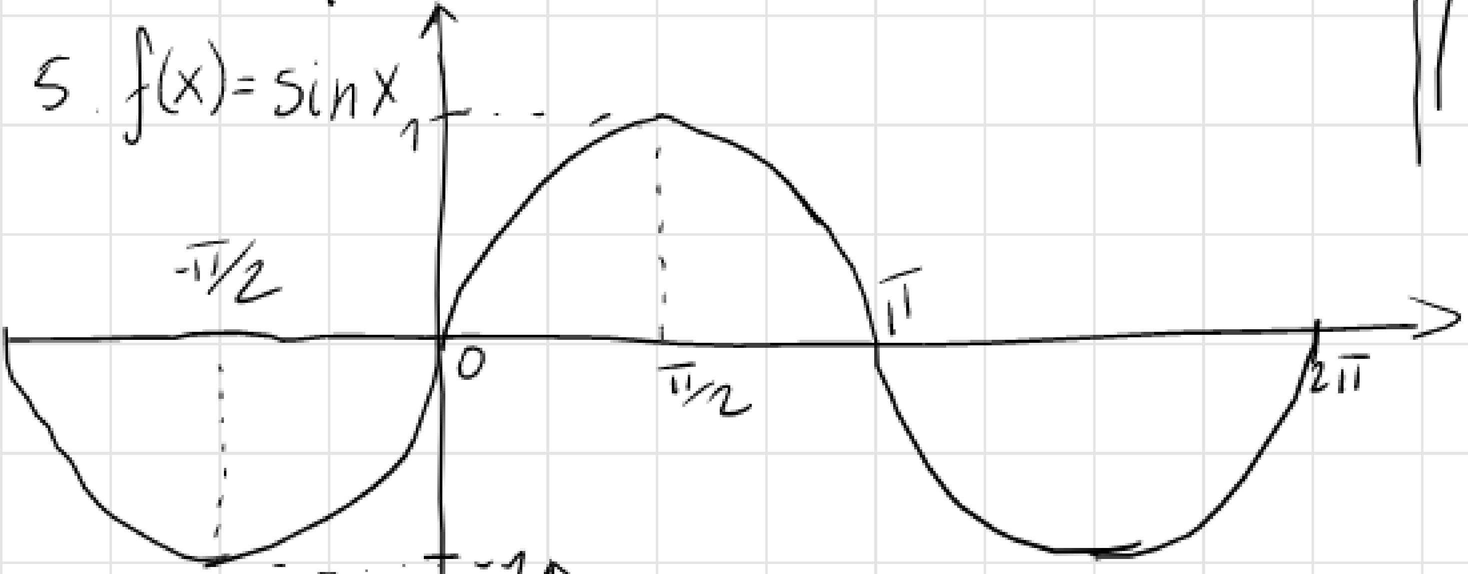


4. $\log_a x = \frac{\ln x}{\ln a} = \frac{1}{\ln a} \cdot \ln x$ ($\ln a \to const$)

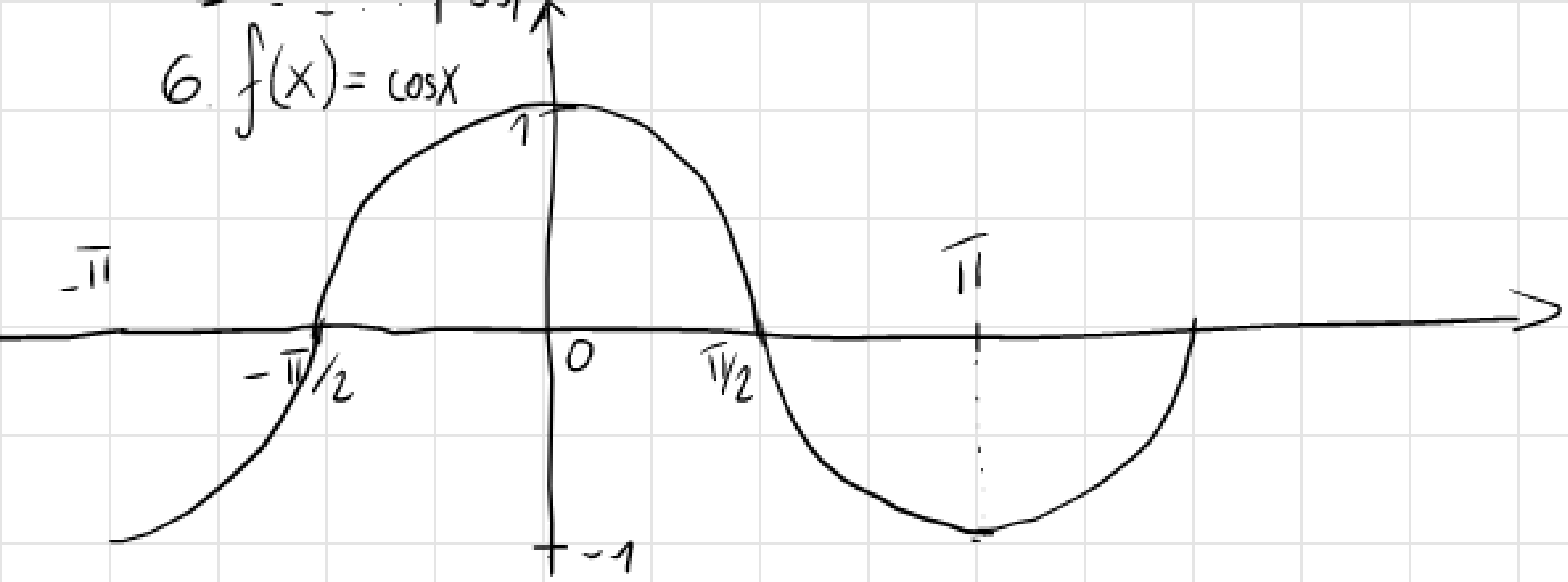
$f(x) = \ln(x)$



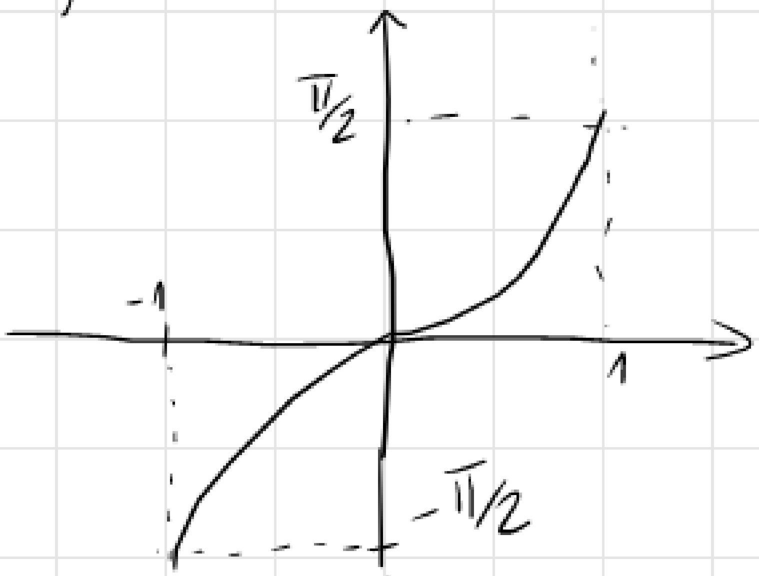
5. $f(x) = \sin x$



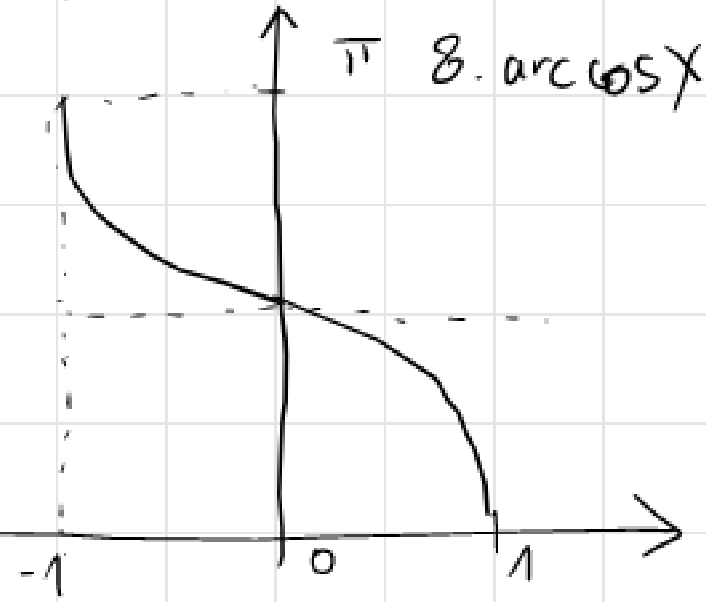
6. $f(x) = \cos x$



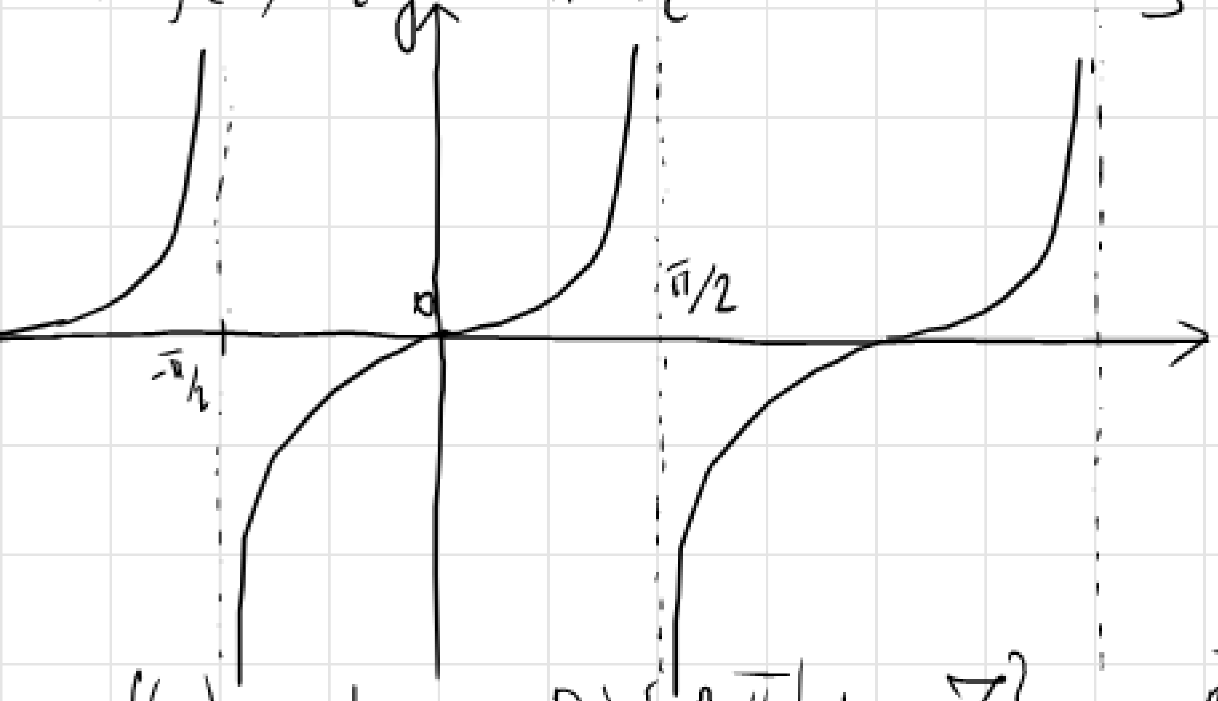
7. $f(x) = \arcsin x$



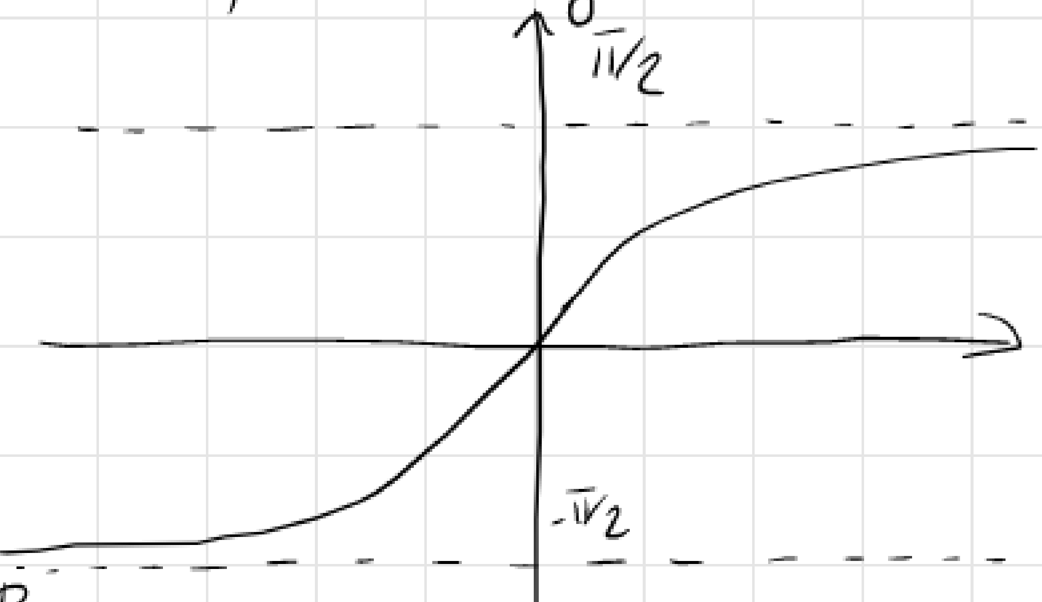
8. $\arccos x$



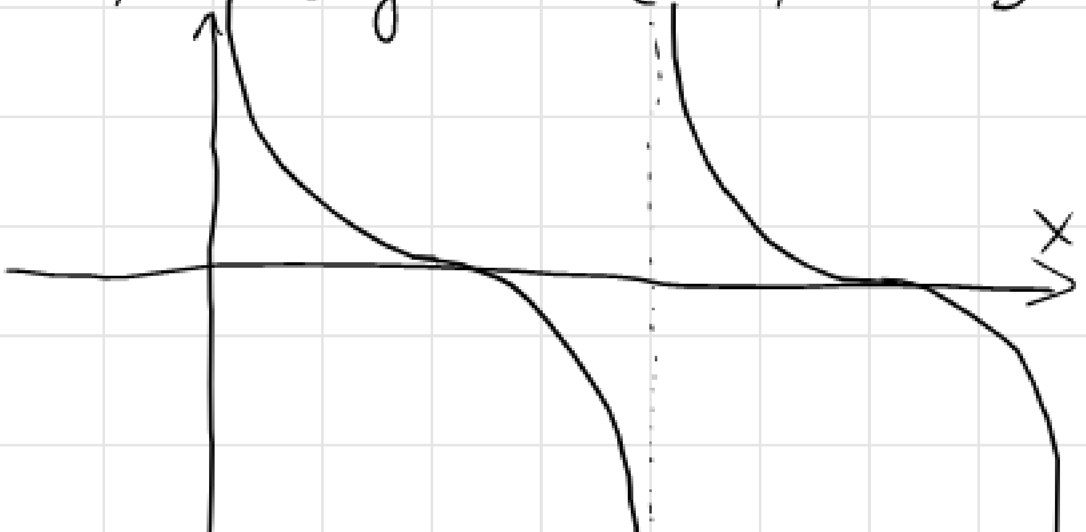
9. $f(x) = tg\,x : \mathbb{R} \setminus \{ \frac{2k+1}{2} \pi \mid k \in \mathbb{Z} \}$



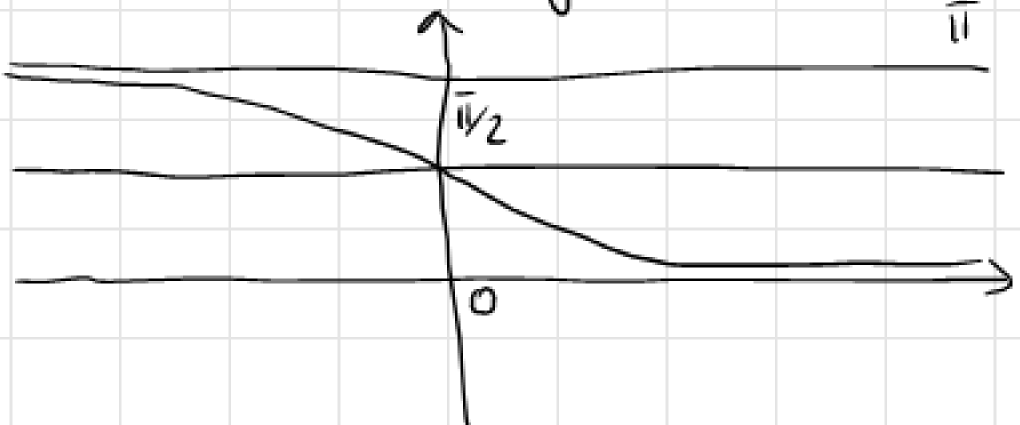
10. $f(x) = arc\,tg(x)$



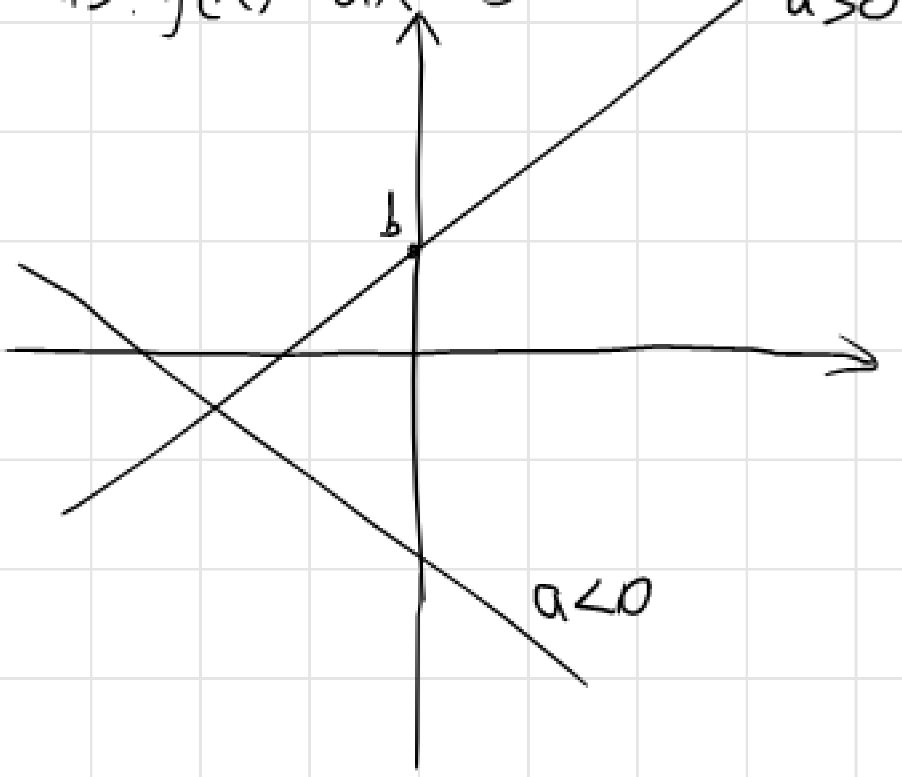
11. $f(x) = ctg\,x \quad R \setminus \{ k\pi \mid k \in \mathbb{Z} \} \to \mathbb{R}$



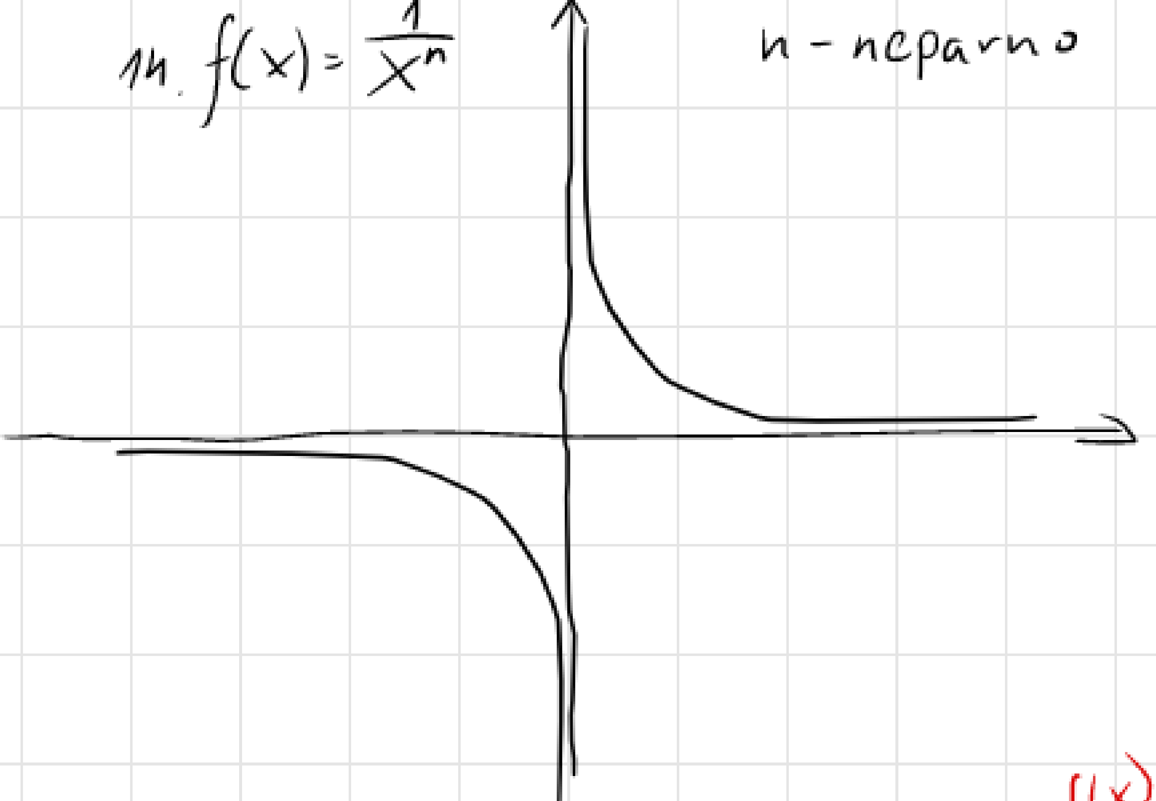
12. $f(x) = arc\,ctg\,x$



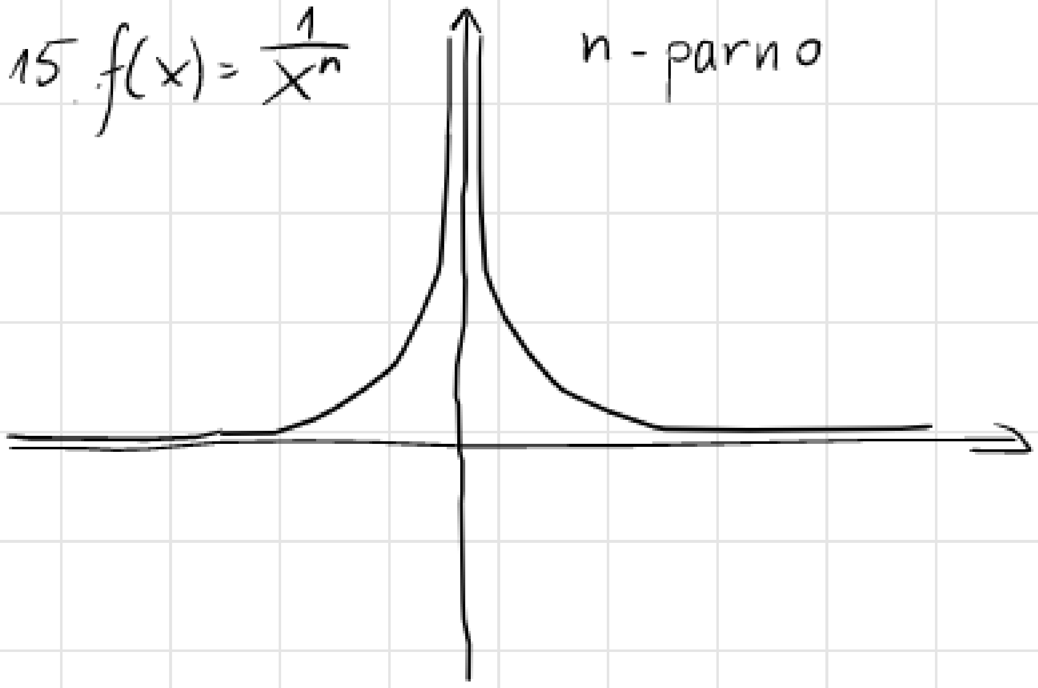
13. $f(x) = ax + b$

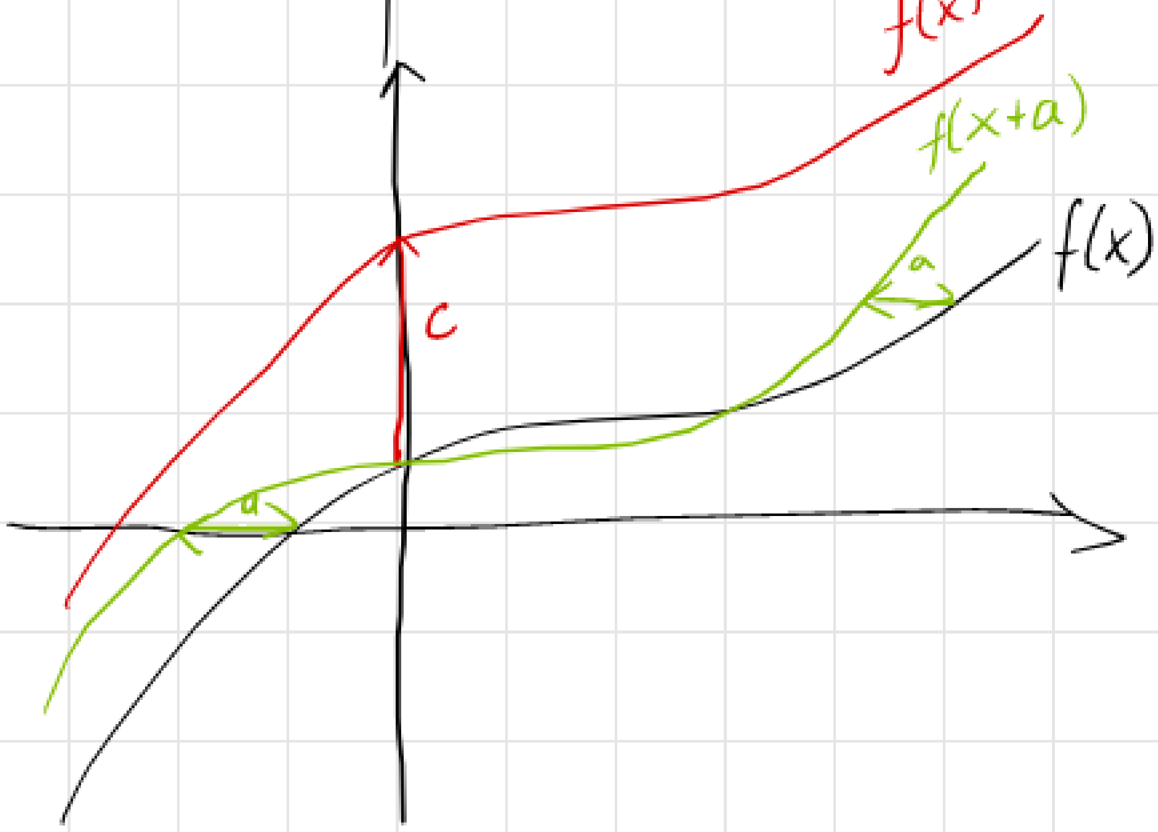


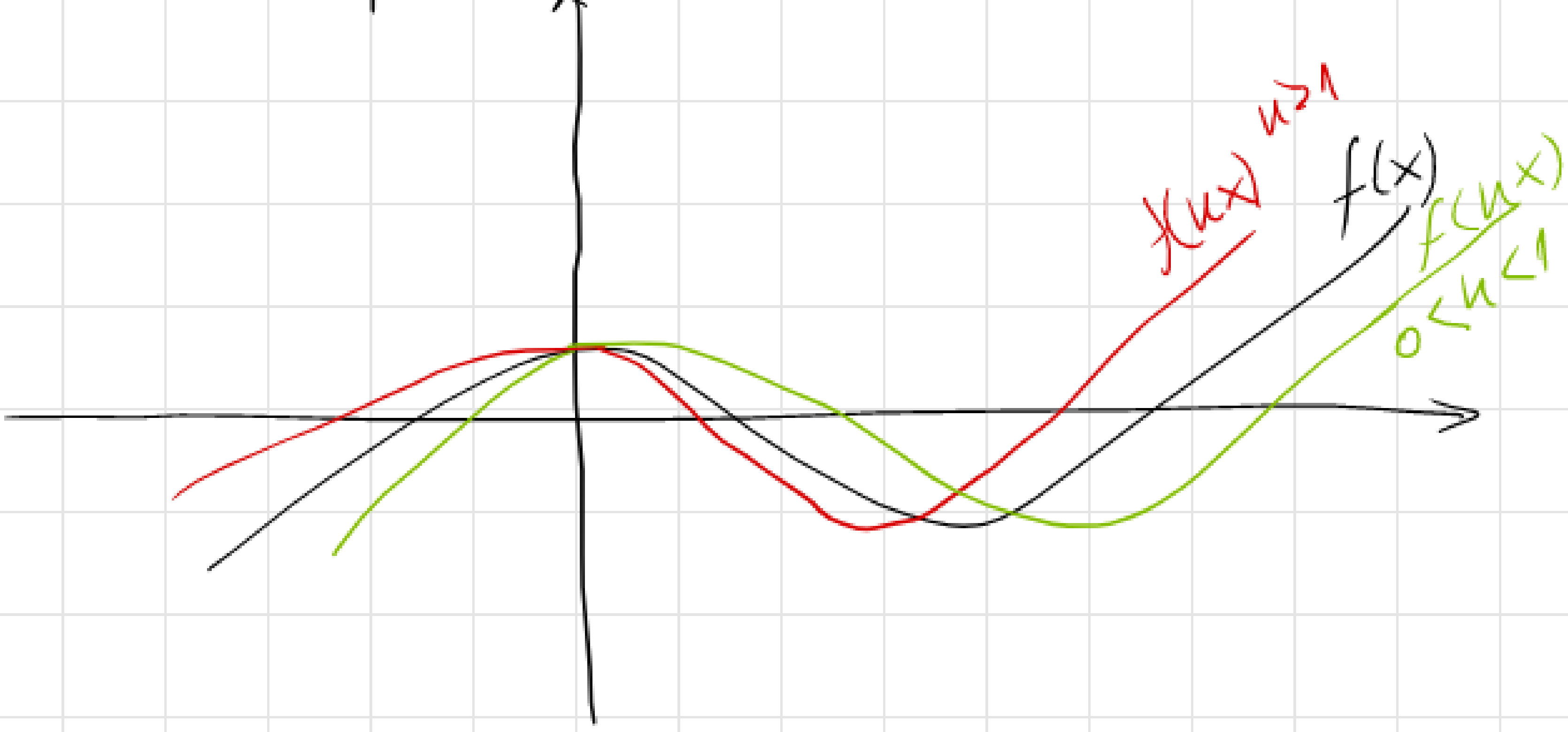
14. $f(x) = \frac{1}{x^n}$ n - neparno



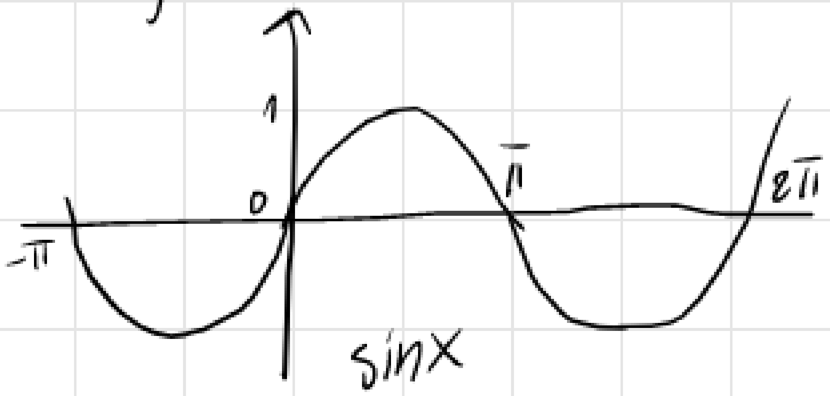
15. $f(x) = \frac{1}{x^n}$ n - parno

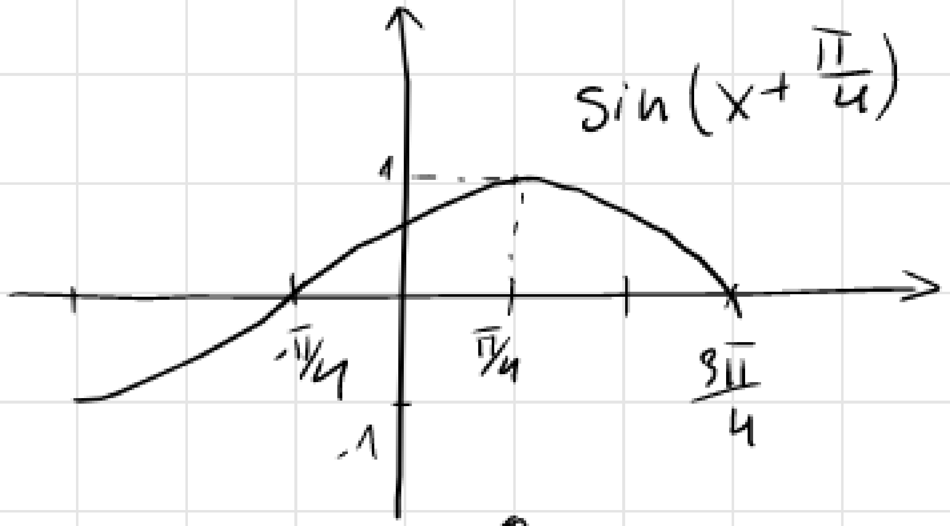


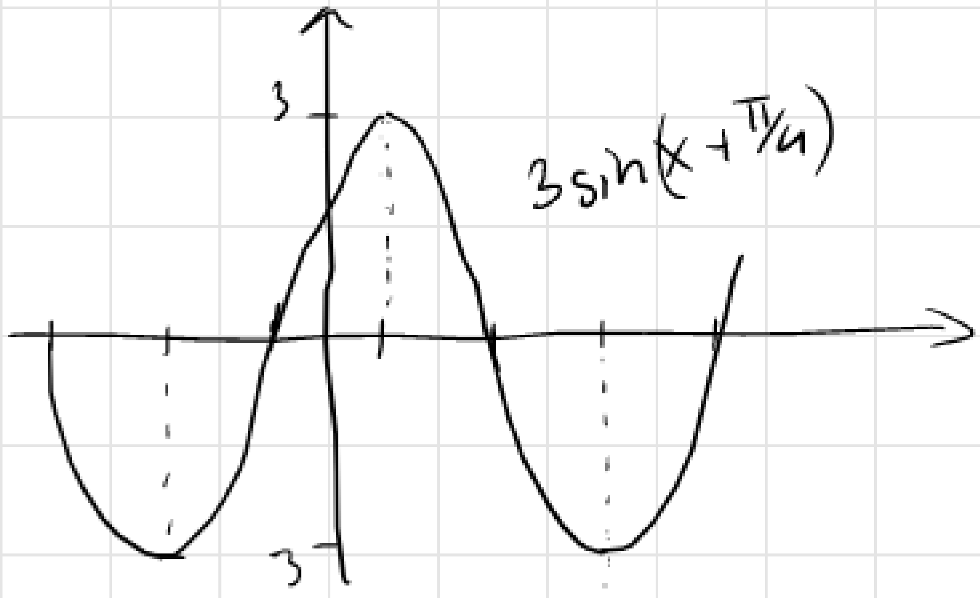


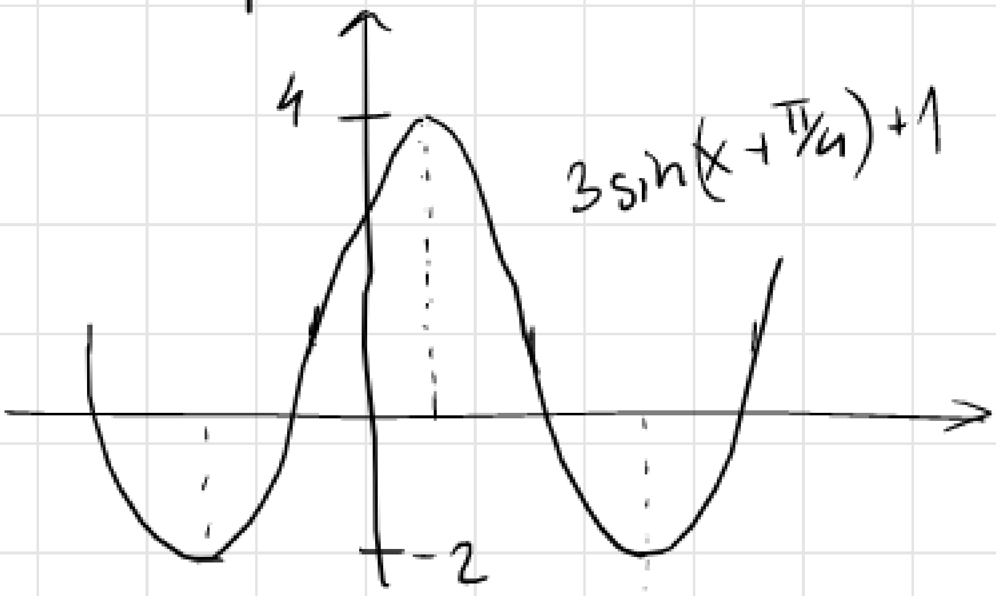


✱ $f(x) = 3\sin(x + \pi/4) + 1$









Matematička indukcija

P(n) Baza: P(1) Hipoteza: Pretpostavimo da važi P(n)

Korak: $P(n) \Rightarrow P(n+1)$

Totalna indukcija: $P(n), P(n-1), \ldots, P(1)$

1. $1+2+\ldots+n = \frac{n(n+1)}{2}$

(B) $1 = \frac{1 \cdot 2}{2} = 1$ ✓ (H) važi $1+2+\ldots+n = \frac{n(1+n)}{2}$

(K) $1+2+\ldots+n+(n+1) = \frac{n(n+1)}{2} + (n+1) = \frac{(n+1)(n+2)}{2}$ ✓

2. $1^2+2^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}$

(B) $1^2 = \frac{1 \cdot 2 \cdot 3}{6} = 1$ ✓ (H) važi $1^2+2^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}$

(K) $n \Rightarrow n+1$

$$1^2+2^2+\ldots+n^2+(n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1) = \frac{(n+1)}{6}\left(n(2n+1)+6n+6\right)$$

$$= \frac{n+1}{6}\left(2n^2+7n+6\right) = \frac{(n+1)(n+2)(2n+3)}{6}$$ ✓

3. $2^n \mid A_n$ ali $2^{n+1} \nmid A_n$

$A_n = (n+1)(n+2) \dots (n+n)$

(B) $n=1$ $A_n = 2$ $2^1 \mid 2$ ✓ $4 \nmid 2$ ✓ Ⓣ (H) važi

(korak) $A_{n+1} = (n+1+1)(n+1+2) \dots (n+1+n-1)(n+1+n)(2n+2)$

$A_{n+1} = \underline{(n+2)(n+3) \dots (n+n)}(n+n+1) \cdot 2 \cdot \underline{(n+1)}$

$A_{n+1} = A_n \cdot 2(2n+1)$

$2^n \mid A_n \Rightarrow 2^{n+1} \mid 2 \cdot A_n \cdot (2n+1) = A_{n+1}$

$2^{n+1} \nmid A_n \Rightarrow 2^{n+2} \nmid 2A_n \Rightarrow 2^{n+2} \nmid 2A_n(2n+1) = A_{n+1}$

4. $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n}} > 2(\sqrt{n+1} - 1)$

B) $1 > 2(\sqrt{2} - 1)$ $1 > 2\sqrt{2} - 2$ $3 > 2\sqrt{2} \,/^2$ $9 > 8$ Ⓣ

(H) važi

(K) $\underbrace{\frac{1}{\sqrt{1}} + \dots + \frac{1}{\sqrt{n}}} + \frac{1}{\sqrt{n+1}} > 2(\sqrt{n+1} - 1) + \frac{1}{\sqrt{n+1}}$

$$2(\sqrt{n+1} - 1) + \frac{1}{\sqrt{n+1}} \overset{?}{>} 2(\sqrt{n+2} - 1)$$

$$2\sqrt{n+1} - \cancel{2} + \frac{1}{\sqrt{n+1}} > 2\sqrt{n+2} - \cancel{2} \quad /^2$$

$$4n+4 + 2 \cdot 2 \cdot \frac{\cancel{\sqrt{n+1}}}{\cancel{\sqrt{n+1}}} + \frac{1}{n+1} > 4n+8$$

$$4n+8 + \frac{1}{n+1} > 4n+8$$

$$\frac{1}{n+1} > 0 \quad ✓$$

5. $n \geq 2$ $\frac{4^n}{n+1} < \frac{(2n)!}{(n!)^2}$ (B) $n=2$ $\frac{\cancel{16}^{2}}{3} < \frac{2 \cdot 3 \cdot 4}{4}$ $\frac{8}{12} < \frac{9}{12}$ ✓

(H) važi (KORAK) $\frac{4^{n+1}}{n+2} < \frac{(2n+2)!}{(n+1)!^2} \rightarrow \frac{(2n)!}{(n!)^2} \cdot \frac{(2n+1)(2n+2)}{(n+1)^2}$

(H) $\Rightarrow \frac{4^n}{\cancel{(n+1)}} \frac{(2n+1)\cancel{(n+1)} \cdot 2}{(n+1)^2} = 4^n \cdot 2 \frac{2n+1}{(n+1)2}$

$\frac{\cancel{4^{n+1}}2}{n+2} \overset{?}{<} \frac{\cancel{4^n} \cdot \cancel{2}(2n+1)}{(n+1)^2}$ $2(n+1)^2 < (n+2)(2n+1)$

$\cancel{2n^2} + 4n + \cancel{2} < \cancel{2n^2} + 5n + \cancel{2}$ $n > 0$ ✓ Ⓣ

2. čas

1. Dokazati $(\forall n \in \mathbb{N})\ 7 \mid 2^{n+1} + 3^{2n-1}$

$(B)_{n=1}$ $2^2+3^{2-1} = 4+3=7$ $7 \mid 7$ Ⓣ (H) važi

Korak: $7 \mid 2^{n+2} + 3^{2n+1}$?

$$2 \cdot 2^{n+1} + 3^{2+2n-1} = 2 \cdot 2^{n+1} + 9 \cdot 3^{2n-1} = 2 \cdot 2^{n+1} + 2 \cdot 3^{2n-1} + 7 \cdot 3^{2n-1} = 2\overbrace{(2^{n+1}+3^{2n-1})}^{(H)7} + \overset{7 \mid \dots}{7 \cdot 3^{2n-1}}$$

II način $2 \to 4 \to 1 \to 2 \to 4 \to 1$

$3 \to 2 \to -1 \to 4 \to 5 \to 1$

$n \equiv 1 \pmod 6$ $2^{n+1} \equiv 4 \pmod 7$ $3^{2n-1} \equiv 3 \pmod 7$

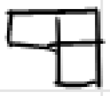
$n \equiv 2 \pmod 6$ $2^{n+1} \equiv 2^2 \equiv \boxed{1} \pmod 7$ $3^{2n-1} \equiv 3^3 \equiv \boxed{-1} \pmod 7$

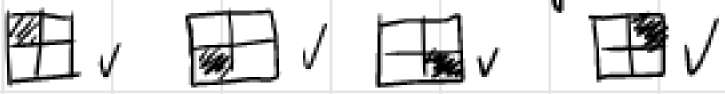
2) $(\forall n \in \mathbb{N})\ 133 \mid 11^{n+2} + 12^{2n+1}$

$(B)_{n=1}$ $11^3 + 12^3 = 3059 \to 133 \mid 3059$ ✓

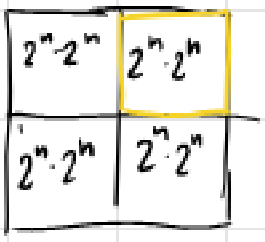
(KORAK) $n \mapsto n+1$ $11^{n+3} + 12^{2n+3} = 11 \cdot 11^{n+2} + 12^2 \cdot 12^{2n+1} = 11 \cdot 11^{n+2} + (133+11) \cdot 12^{2n+1}$

$= 11(11^{n+2} + 12^{2n+1}) + 133 \cdot 12^{2n+1} \xRightarrow{(ih)} 133 \mid (11^{n+2}+12^{2n+1})$ i $133 \mid 133 \cdot 12^{2n+1}$ ✓

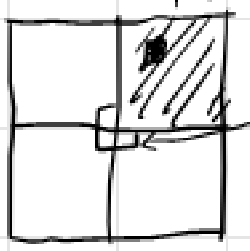
3) Dokazati da se tabla $2^n \times 2^n$ bez 1 polja može popločati sa  bez preklapanja.

Baza: $n=1$ 2×2 

Korak: $2^n \times 2^n$ možemo $\longmapsto 2^{n+1} \times 2^{n+1}$?



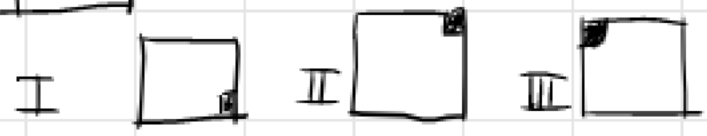
prazno polje će se naći u jednom od ova četiri. B.G.O. neka je to žuti. Po (ih) znamo da može da se poploča



popločano

dodamo figuru „na ćošak"

Tada nam između 3 table $2^n \times 2^n$ i svaka od njih što zauzima jedno polje.



Po pretpostavci svaka od te 3 table bez 1 polja može popločati.

4) $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$ $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ $0! = 1$

Baza: $L = (a+b)^1 = a+b$ $D = \sum_{k=0}^{1} \binom{1}{k} a^{1-k} b^k = \binom{1}{0} a \cdot 1 + \binom{1}{1} 1 \cdot b = a+b$ Ⓣ ✓

Korak: $n \mapsto n+1$

$$(a+b)^{n+1} = (a+b)(a+b)^n = a(a+b)^n + b(a+b)^n \overset{(ih)}{=} a\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k + b\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k =$$

$$= \sum_{k=0}^{n} \binom{n}{k} a^{n-k+1} b + \sum_{k=0}^{n} \binom{n}{\ell} \cdot a^{n-\ell} \cdot b^{\ell+1}$$

Kada je: $\left.\begin{matrix} n+1-k = n-\ell \\ k = \ell+1 \end{matrix}\right\}$ $k = \ell+1$, $k=0$, $\ell = n$

$$\overset{*}{=} \sum_{k=1}^{n} \left(\binom{n}{k} + \binom{n}{k-1}\right) a^{n+1-k} b^k + \binom{n}{0} a^{n+1} + \binom{n}{n} b^{n+1} =$$

$$\overset{*}{=} \sum_{k=1}^{n} \binom{n+1}{k} a^{n+1-k} \cdot b^k + \binom{n+1}{0} a^{n+1} + \binom{n+1}{n+1} b^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} a^{n+1-k} b^k$$

$$* \binom{n}{k} + \binom{n}{k-1} = \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k-1)!}$$

$$= \frac{n(n-1)\cdot\ldots\cdot(n-k+1)}{k!} + \frac{n(n-1)\cdot\ldots\cdot(n-k+2)}{(k-1)!}$$

$$= \frac{n(n-1)\cdots(n-k+2)}{(k-1)!}\left(\frac{n-k+1}{k} + \frac{k}{k}\right)$$

$$= \binom{n+1}{k}$$

Totalna indukcija

$P(n)$ Baza: $P(1)$ Korak: $P(1), P(2), P(3), \dots, P(n) \Leftarrow\Rightarrow P(n+1)$

1. $a_0 = 1$ $a_n = a_{n-1} + a_{n-2} + \dots + a_2 + a_1 + 2a_0 \Rightarrow a_n = 2^n$

(B) $n=1$ $a_1 = 2a_0 = 2 = 2^1 = 2$ Ⓣ

Korak: $a_1 = 2^1$ $a_2 = 2^2$ $a_3 = 2^3 \dots a_n = 2^n \Rightarrow a_{n+1} = 2^{n+1}$?

$$a_{n+1} = a_n + a_{n-1} + \dots + a_2 + a_1 + 2a_0 = 2^n + 2^{n-1} + \dots + 2^1 + 1 + 1 = 2^n + 2^{n-1} + \dots + 2 + 1 + 1$$

$$= \frac{2^{n+1} - 1}{2 - 1} + 1 = 2^{n+1} - 1 + 1 = 2^{n+1}$$

Nejednakosti

Bernulijeva jednakost

$x > -1$, $n \in \mathbb{N}$; $(1+x)^n \geq 1 + n \cdot x$

(B) $n=1$ $1 + x \geq 1 + x$ ✓ (Korak) $(1+x)^{n+1} = (1+x)(1+x)^n \overset{(ih)}{\geq} (1+x)(1+n\cdot x) = 1 + nx + x + nx^2$

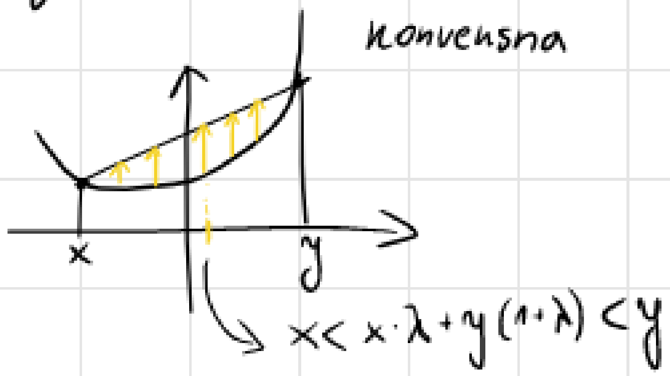
$= 1 + (n+1)x + x^2$

$\geq 1 + (n+1)x$

* $a^2 + b^2 \geq 2ab$ $a^2 + b^2 - 2ab = (a-b)^2 \geq 0 \Rightarrow a^2 + b^2 \geq 2ab$

$a^2 + b^2 + c^2 \geq ab + bc + ca \;/\cdot 2$ $2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca \geq 0$

$(a-b)^2 + (a-c)^2 + (b-c)^2 \geq 0$ ✓



Def $f : [a,b] \to \mathbb{R}$. f je konveksna ako:

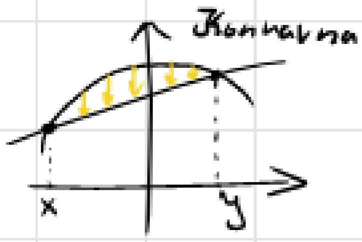
$(\forall x, y \in [a,b])\,(\forall \lambda \in [0,1])$

$\lambda f(x) + (1-\lambda) f(y) \geq f(\lambda x + (1-\lambda) y)$

Konkavna $\lambda f(x) + (1-\lambda) f(y) \leq f(\lambda x + (1-\lambda) y)$

$\Updownarrow$ $(\forall x, y, z \in [a,b])\; x < z < y$

$\Rightarrow \frac{y-z}{y-x} f(x) + \frac{z-x}{y-x} f(y) \geq f(z)$



Jensenova nejednakost

$f : [a,b] \longrightarrow R$ konveksna $\lambda_1, \lambda_2, \dots, \lambda_n > 0$ $\lambda_1 + \lambda_2 + \dots + \lambda_n = 1$ $x_1, x_2, \dots, x_n \in [a,b]$

$\Rightarrow \lambda_1 f(x_1) + \lambda_2 f(x_2) + \dots + \lambda_n f(x_n) \geq f(\lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_n x_n)$

Dokaz: (B) $n=1$ $1 \cdot f(x_1) \geq f(1 \cdot x_1)$ ✓ $n=2$ $\lambda_1 \in (0,1)$ $\lambda_2 = 1 - \lambda_1$ $\lambda_1 + \lambda_2 = 1$

$\lambda_1 f(x_1) + \lambda_2 f(x_2) \geq f(\lambda_1 x_1 + \lambda_2 x_2)$

Korak: $n \longmapsto n+1$ $\lambda_1 + \lambda_2 + \dots + \lambda_{n+1} = 1$ $x_i \in [a,b]$ $i \in \{1, \dots, n\}$

$\lambda_1 f(x_1) + \lambda_2 f(x_2) + \dots + \lambda_n f(x_n) + \lambda_{n+1} f(x_n)$ $\lambda_1 + \lambda_2 + \dots + \lambda_n = 1 - \lambda_{n+1}$

$(1 - \lambda_{n+1})\left(\frac{\lambda_1}{1-\lambda_{n+1}} f(x_1) + \dots + \frac{\lambda_n}{1-\lambda_{n+1}} f(x_n) + \lambda_{n+1} f(x_{n+1})\right)$

$\downarrow$

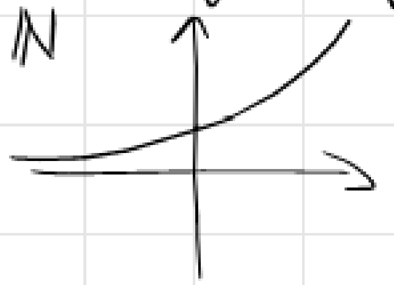
$$\overset{(ih)}{\geq} (1-\lambda_{n+1}) f\left(\frac{\lambda_1}{1-\lambda_{n+1}} x_1 + \ldots + \frac{\lambda_n}{1-\lambda_{n+1}} x_n\right) + \lambda_{n+1} f(x_{n+1})$$

$$\overset{konv.}{\geq} f\left((1-\lambda_{n+1})\left(\frac{\lambda_1}{1-\lambda_{n+1}} x_1 + \ldots + \frac{\lambda_n}{1-\lambda_{n+1}} x_n\right) + \lambda_{n+1} x_{n+1}\right)$$

$$= f(\lambda_1 x_1 + \ldots + \lambda_{n+1} x_{n+1}) \quad \omega$$

AG: $\frac{a_1 + \ldots + a_n}{n} \geq \sqrt[n]{a_1 \cdot \ldots \cdot a_n}$ Nejednaost aritmeticke i geometrijske sredine.

$a_1, \ldots, a_n > 0$, $n \in \mathbb{N}$



$f(x) = e^x$ eksponencijalna konveksna

$a_1, a_2, \ldots, a_n > 0 \longmapsto x_1 = \ln a_1, \ldots\ldots x_n = \ln a_n$

$\lambda_1 = \lambda_2 = \ldots = \lambda_n = \frac{1}{n}$ Jensen

$$\frac{1}{n} f(x_1) + \frac{1}{n} f(x_2) + \ldots + \frac{1}{n} f(x_n) \geq f\left(\frac{x_1 + x_2 + \ldots x_n}{n}\right)$$

$$\frac{1}{n} e^{x_1} + \frac{1}{n} e^{x_2} + \ldots + \frac{1}{n} e^{x_n} \geq e^{1\left(\frac{\ln a_1 + \ldots + \ln a_n}{n}\right)} = \ln(a_1 \cdot \ldots \cdot a_n)$$

$$\frac{a_1 + a_2 + \ldots + a_n}{n} \geq (a_1 \cdot a_2 \cdot a_3 \cdot \ldots \cdot a_n)^{\frac{1}{n}} = \sqrt[n]{a_1 \cdot a_2 \cdot \ldots \cdot a_n}$$

Kvadratna: $\sqrt{\frac{a_1^2 + \ldots + a_n^2}{n}}$ $\quad K \geq A \geq G$

$f(x) = x^2$ konveksna $\quad a_1, a_2, \ldots, a_n \in \mathbb{R} \quad \lambda_1 = \lambda_2 = \ldots = \lambda_n = \frac{1}{n}$

$$\frac{1}{n} f(a_1) + \frac{1}{n} f(a_2) + \ldots + \frac{1}{n} f(a_n) \overset{Jensen}{\geq} f\left(\frac{a_1 + \ldots + a_n}{n}\right)$$

$$\frac{a_1^2 + \ldots + a_n^2}{n} \geq \left(\frac{a_1 + \ldots + a_n}{n}\right)^2 \Rightarrow \sqrt{\frac{a_1^2 + \ldots + a_n^2}{n}} \geq \frac{a_1 + \ldots + a_n}{n}$$

$x, y \in \mathbb{R}$, $a \in (0,1)$?

$$\lambda f(x) + (1-\lambda) f(y) \geq f(\lambda x + (1-\lambda) y) \iff \lambda x^2 + (1-\lambda) y^2 \geq (\lambda x + (1-\lambda) y)^2 =$$

$$= \lambda^2 x^2 + (1-\lambda)^2 y^2 + 2\lambda(1-\lambda) x y$$

$$\iff \lambda(1-\lambda) x^2 + (1-\lambda)\lambda y^2 \geq 2\lambda(1-\lambda) x y \quad / : (1-\lambda)\lambda$$

$$\iff x^2 + y^2 \geq 2xy \quad Ⓣ \qquad K_n \geq A_n \geq G_n \geq H_n \longrightarrow \text{Domaci}$$

Harmonijska $\frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}} = H_n$

$$* \; 1 \cdot 2^2 \cdot 3^3 \cdot \ldots \cdot n^n \leq \left(\frac{2n+1}{3}\right)^{\left(\frac{n(n+1)}{2}\right)}$$

$\frac{n(n+1)}{2}$ članova

$$1 \cdot \underbrace{2 \cdot 2}_{2\ puta} \cdot \underbrace{3 \cdot 3 \cdot 3}_{3\ puta} \cdot \ldots \cdot \underbrace{n \cdot n \cdot \ldots \cdot n}_{n\ puta} \leq \frac{1 + 2 \cdot 2 + 3 \cdot 3 + \ldots + n^2}{\frac{n(n+1)}{2}} = \frac{\frac{\cancel{n(n+1)}(2n+1)}{\cancel{6}\,3}}{\frac{\cancel{n(n+1)}}{\cancel{2}}} = \frac{(2n+1)}{3}$$

Nejednaost trougla: $|x+y| \leq |x| + |y|$ ← za $x, y \in \mathbb{R}$

Koši-Švarcova nejednaost $(a_1 b_1 + a_2 b_2 + \ldots + a_n b_n)^2 \leq (a_1^2 + \ldots + a_n^2)(b_1^2 + \ldots + b_n^2)$

Digresija, vežbe sa početna

$\sin \frac{n\pi}{2} = \begin{cases} 1, & n \equiv 1 \pmod 4 \\ -1, & n \equiv 3 \pmod 4 \\ 0; & 2 \mid n \end{cases}$

$\frac{n^2-9}{5n^2+2} \longleftarrow n \equiv 1 \pmod 4$

$\frac{1}{5} - \frac{9+2/5}{5n^2+2} \nearrow \quad n = 4k+1$

* $x > 0,\ x \in \mathbb{R}$

$A = \left\{ \frac{\lfloor nx \rfloor}{n} \,\middle|\, n \in \mathbb{N} \right\}$ $\sup A = x?$ $\lfloor nx \rfloor \leq nx \Rightarrow \frac{\lfloor nx \rfloor}{n} \leq x \Rightarrow A \leq x$

$(\forall \varepsilon > 0)(\exists a \in A)\ a > x - \varepsilon$ $\quad \frac{\lfloor nx \rfloor}{n} > x - \varepsilon$

$\Leftrightarrow \lfloor nx \rfloor > nx - n\varepsilon$

$\Leftrightarrow nx - \underset{\ulcorner 1}{\lfloor nx \rfloor} < n\varepsilon \quad n\varepsilon \geq 1 \quad n \geq \frac{1}{\varepsilon}$

$n_0 = \left\lfloor \frac{1}{\varepsilon} \right\rfloor + 1 \quad n_0 \geq \frac{1}{\varepsilon} \Rightarrow n_0 \varepsilon \geq 1 \Rightarrow n_0 x - \lfloor n_0 x \rfloor \Rightarrow n_0 \varepsilon > n_0 x - \lfloor n_0 x \rfloor$

$\Rightarrow \frac{\lfloor n_0 x \rfloor}{n_0} > x - \varepsilon$

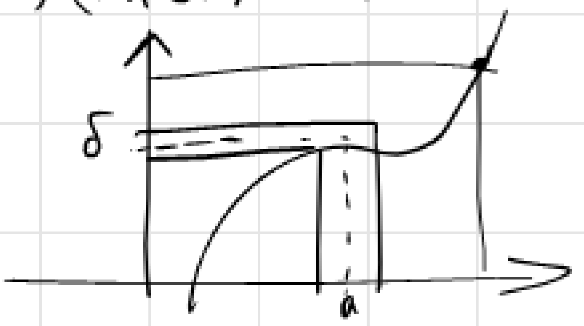
Dakle $(\forall \varepsilon > 0)(\exists a \in A)\ a > x - \varepsilon$

I i II $\sup A = x$

3. čas

$f: A \to \mathbb{R}$ a, tačka nagomilavanja skupa A

$\lim_{x\to a} f(x) = b \overset{def}{\Longleftrightarrow} (\forall \varepsilon>0)(\exists \delta>0)(\forall x\in A)\ 0<|x-a|<\delta \Rightarrow |f(x)-b|<\varepsilon$



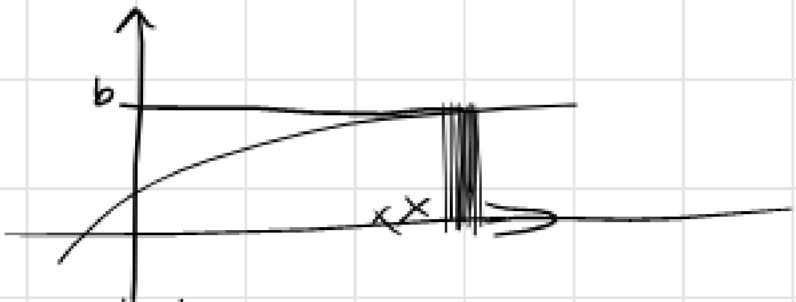
$\lim_{x\to a} f(x) = +\infty$
$\Updownarrow$

$(\forall M>0)(\exists \delta>0)(\forall x\in A)\ 0<|x-a|<\delta \Rightarrow f(x)>M$

$f(x)<-M$

$+\infty$ tačka nagomilavanja A

$\lim_{x\to +\infty} f(x) = b$



$(\forall \varepsilon>0)(\exists M>0)(x\in A)\ x>M \Rightarrow |f(x)-b|<\varepsilon$

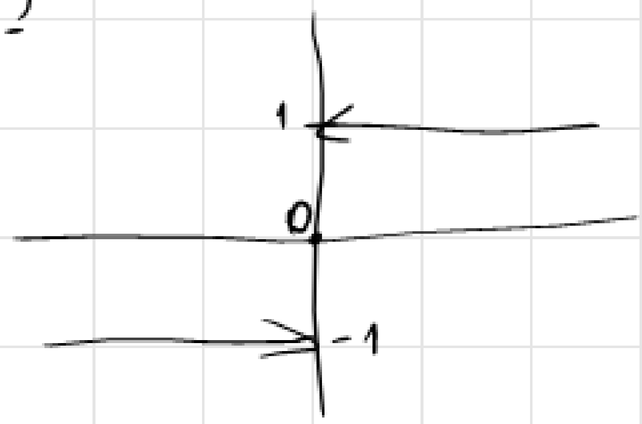
$\lim_{x\to +\infty} f(x) = +\infty$
$(\forall M>0)(\exists M>0)(\forall x\in A)\ \ x>M \Rightarrow f(x)>M$

$\lim_{x\to a+} f(x) = b \Longleftrightarrow (\forall \varepsilon>0)(\exists \delta>0)(\forall x\in A)\ a<x<a+\delta \Rightarrow |f(x)-b|<\varepsilon$

$a-b<x<a$

$\lim_{x\to a} f(x) = b \Longleftrightarrow \lim_{x\to a+} f(x) = b \wedge \lim_{x\to a^-} f(x) = b$

$f(x) = sgn(x)$ $f(x) = \begin{cases} 1, & x>0 \\ 0, & x=0 \\ -1, & x<0 \end{cases}$



$\lim_{x\to 0+} f(x) = 1$

$(\forall \varepsilon>0)(\exists \delta>0)\ 0<x<\delta \Rightarrow |f(x)-1|<\varepsilon ?$

$\varepsilon>0$ proizvoljno $\delta=1$ $0<x<1 \Rightarrow f(x)=1 \Rightarrow |f(x)-1| = 0 < \varepsilon$

① $f(x) = 3x+5$ $\lim_{x\to 0} f(x) = 5$

$\varepsilon>0$ -proizvoljno $?\delta>0?$ - tražim $0<|x|<\delta$

$\Rightarrow |3x+5-5|<\varepsilon$ $|x|<\frac{\varepsilon}{3}$

$\delta = \varepsilon/3 > 0$

$|3x+5-5| = |3x| = 3|x| < 3\delta = \varepsilon$

! Važe limesi:

1. $\lim\limits_{x\to 0}\frac{\sin x}{x}=1$ 2. $\lim\limits_{x\to 0}\frac{1-\cos x}{x^2}=\frac{1}{2}$ 3. $\lim\limits_{x\to +\infty}\left(1+\frac{1}{x}\right)^x=e$

4. $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}}=e$ 5. $\lim\limits_{x\to 0}\frac{e^x-1}{x}=1$ 6. $\lim\limits_{x\to 0}\frac{\ln(1+x)}{x}=1$

7. $\lim\limits_{x\to 0}\frac{(1+x)^\alpha-1}{x}=\alpha$

$\lim\frac{e^x-1}{x}=1$

* $$\lim_{x\to 0}\frac{(1+x)^\alpha-1}{x}=\lim_{x\to 0}\frac{e^{\alpha\ln(1+x)}-1}{x}=\lim_{x\to 0}\frac{e^{\alpha\ln(1+x)}-1}{\alpha\ln(1+x)}\cdot\frac{\alpha\cdot\ln(1+x)}{x}=$$

$$=\lim_{x\to 0}\alpha\overset{1}{\boxed{\frac{\ln(1+x)}{x}}}=\alpha$$

$\lim\limits_{x\to 0}\frac{\ln(1+x)}{x}=1$

2.* $\lim\limits_{x\to 0}\frac{\cos x}{x}=?=\frac{1}{0}?$ $\underbrace{\lim\limits_{x\to 0^+}\frac{\cos x}{x}=+\infty \quad \lim\limits_{x\to 0^-}\frac{\cos x}{x}=-\infty}_{\text{ne postoji } \lim\limits_{x\to 0}\frac{\cos x}{x}}$

3. $$\lim_{x\to 0}\frac{\sin 5x-\sin 3x}{\sin x}=\lim_{x\to 0}\frac{\overset{5}{\boxed{\frac{\sin 5x}{x}}}-\overset{3}{\boxed{\frac{\sin 3x}{x}}}}{\boxed{\frac{\sin x}{x}}\to 1}=5-3=2$$

4. $$\lim_{x\to a}\frac{\sin x-\sin a}{x-a}=\lim_{x\to a}\frac{2\sin\frac{x-a}{2}\cos\frac{x+a}{2}}{x-a}=\lim_{x\to a}\overset{1}{\boxed{\frac{\cancel{2}\sin\frac{x-a}{2}}{\cancel{2}\frac{x-a}{2}}}}\cdot\cos\frac{x+a}{2}=$$

$$=\lim_{x\to a}\cos\frac{x+a}{2}\overset{x=a+t}{=}\lim_{t\to 0}\cos\left(a+\frac{t}{2}\right)=\lim_{t\to 0}\left(\cos a\overset{1}{\boxed{\cos\frac{t}{2}}}-\sin a\overset{0}{\boxed{\sin\frac{t}{2}}}\right)$$

$$=\boxed{\cos a}$$

5. $$\lim_{x\to 5}\frac{tg(\pi\cdot x)}{x-\sqrt{7}}=\frac{\overset{tg(5\pi)=0}{0}}{5-\sqrt{7}}=0$$

6. $$\lim_{x\to 0}\frac{e^{ax}-e^{bx}}{x}=\lim_{x\to 0}\frac{e^{ax}-1-(e^{bx}-1)}{x}=\lim_{x\to 0}\left(\frac{e^{ax}-1}{x}-\frac{e^{bx}-1}{x}\right)=$$

$$=\lim_{x\to 0}\left(\frac{a\cdot(e^{ax}-1)}{ax}-\frac{b(e^{bx}-1)}{bx}\right)=\boxed{a-b}$$

$$7.\ \lim_{x\to 1}\frac{1-x^2}{\sin \pi x}=\lim_{x\to1}\frac{(1-x)(1+x)}{\sin\pi x}\overset{t=1-x}{=\!=}\lim_{t\to0}\frac{t(2-t)}{\sin(\pi(1-t))}=$$

$$=\lim_{t\to0}(2-t)\frac{t}{\sin(\pi-\pi t)}=\lim_{t\to0}(2-t)\frac{t}{\sin\pi t}=\lim_{t\to0}(2-t)\overset{1}{\boxed{\frac{\pi\cdot t}{\sin\pi t}}}=2\cdot1=2$$

$$8.\ \lim_{x\to+\infty}x(\ln(1+x)-\ln x)=\lim_{x\to+\infty}x\cdot\ln\frac{1+x}{x}=\lim_{x\to+\infty}\ln\overset{e}{\boxed{\left(1+\frac{1}{x}\right)^x}}=\ln e=1$$

$$\overset{t=\frac{1}{x}}{=\!=}\lim_{x\to0^+}\frac{\ln(1+t)}{t}=1$$

$$9.\ \lim_{x\to0}\frac{1}{x}\ln\sqrt{\frac{1+x}{1-x}}=\lim_{x\to0}\frac{\ln\left(\frac{1+x}{1-x}\right)}{2x}=\lim_{x\to0}\frac{\ln\left(1+\frac{2x}{1-x}\right)}{2x}=\lim_{x\to0}\overset{1}{\boxed{\frac{1}{1-x}}}\overset{1}{\boxed{\frac{\ln\left(1+\frac{2x}{1-x}\right)}{\underbrace{\frac{2x}{1-x}}_{\to0}}}}=1$$

$$10.\ \lim_{x\to0}(\cos x)^{\frac{1}{3x^2}}=\lim_{x\to0}e^{\frac{1}{3x^2}\cdot\ln(\cos x)}$$

$$\boxed{*\ f(x)^{g(x)}=e^{\ln f(x)^{g(x)}}=e^{g(x)\ln f(x)}}$$

$$=\lim_{x\to0}e^{\frac{1}{3x^2}\cdot\ln(1+(\cos x-1))}$$

$$\overset{*}{=}\lim_{x\to0}e^{\frac{1}{3x^2}\underbrace{\frac{\ln(1+(\cos x-1))}{\underbrace{(\cos x-1)}_{\to0}}}_{1}\cdot(\cos x-1)}$$

$$=\lim_{x\to0}e^{\overset{*}{}\frac{\cos x-1}{3x^2}}=\lim_{x\to0}e^{-\frac{1}{6}}=e^{-\frac{1}{6}}=\frac{1}{\sqrt[6]{e}}$$

$$*\ \lim_{x\to0}\frac{1-\cos x}{x^2}=\frac{1}{2}$$

$$*=\frac{-\overset{\frac{1}{2}}{\boxed{(1-\cos x)}}}{3\,\boxed{x^2}}=-\frac{1}{3}\cdot\frac{1}{2}=-\frac{1}{6}$$

$$11.\ \lim_{x\to0}\frac{\sqrt[n]{\cos(x\cdot\alpha)}-\sqrt[n]{\cos(\beta x)}}{x^2}=\lim_{x\to0}\frac{(\cos x(\alpha x))^{\frac{1}{n}}-(\cos x(\beta x))^{\frac{1}{n}}}{x^2}=$$

$$=\lim_{x\to0}\frac{(1+(\cos\alpha x-1))^{\frac{1}{n}}-(1+(\cos\beta x-1))^{\frac{1}{n}}}{x^2}=\lim_{x\to0}\frac{(\cos\alpha x-1)\cdot\overset{\frac{1}{n}}{\boxed{\frac{(1+(\cos\alpha x-1))^{\frac{1}{n}}}{(\cos\alpha x-1)}}}-(\cos\beta x-1)\overset{\frac{1}{n}}{\boxed{\frac{(1+(\cos x-1))^{\frac{1}{n}}}{\underbrace{(\cos\beta x-1)}_{0}}}}}{x^2}$$

$$=\lim_{x\to0}\frac{\frac{1}{n}(\cos\alpha x-1)-\frac{1}{n}(\cos\beta x-1)}{x^2}=\lim_{x\to0}\frac{\frac{1}{n}\cdot\alpha^2x^2\overset{-\frac{1}{2}}{\boxed{\frac{(\cos\alpha x-1)}{(\alpha x)^2}}}-\frac{1}{n}\beta^2x^2\cdot\boxed{\frac{(\cos\beta x-1)}{(\beta x)^2}}\to-\frac{1}{2}}{x^2}$$

$$=\lim_{x\to0}\frac{\frac{1}{2}\left(-\frac{\alpha^2}{n}+\frac{\beta^2}{n}\right)x^2}{x^2}=\frac{1}{2}\left(\frac{\beta^2}{n}-\frac{\alpha^2}{n}\right)$$

$$*\ \lim_{x\to+\infty}\frac{a_mx^m+a_{m-1}x^{m-1}+\dots+a_1x+a_0}{b_nx^n+b_{n-1}x^{n-1}+\dots+b_1x+b_0}=\begin{cases}0, & m<n\\ \frac{a_m}{b_n}, & m=n\\ +\infty, & m>n \quad \frac{a_m}{b_n}>0\\ -\infty, & m>n \quad \frac{a_m}{b_n}<0\end{cases}$$

$$*\ \lim_{x\to0}\frac{a_mx^m+a_{m-1}x^{m-1}+\dots+a_1x+a_0}{b_nx^n+b_{n-1}x^{n-1}+\dots+b_1x+b_0}$$ izvlačimo najmanji

$$\lim_{x\to0}\frac{x^2+3x^4+5x}{x^{10}-6x}=\lim_{x\to0}\frac{x(5+x+3x^3)}{\cancel{x}(x^9-6)}=\lim_{x\to0}\frac{5+x+3x^3}{(g^6-6)}=-\frac{5}{6}$$

$$12.\ \lim_{x\to1}\frac{x^m-1}{x^n-1} \overset{x=t+1}{=\!=} \lim_{t\to0}\frac{(t+1)^m-1}{(t+1)^n-1} = \lim_{t\to0}\frac{\cancel{1}+\binom{m}{1}t+\binom{m}{2}t^2+\dots+\binom{m}{m}t-\cancel{1}}{\cancel{1}+\binom{n}{1}t+\binom{n}{2}t^2+\dots+\binom{n}{n}t-\cancel{1}}$$

$$=\lim_{t\to0}\frac{t\left(m+\binom{m}{2}t+\binom{m}{3}t^2+\dots+\binom{m}{m}t^{m-1}\right)}{t\left(n+\binom{n}{2}t+\binom{n}{3}t^2+\dots+\binom{m}{m}t^{n-1}\right)} =\!= \lim_{t\to0}\frac{m+\binom{m}{2}t+\dots+t^{m-1}}{n+\binom{n}{2}t+\dots+t^{n-1}} = \frac{m}{n}$$

$$13.\ \lim_{x\to+\infty}\left(\frac{x+a}{x-a}\right)^x = \lim_{x\to+\infty} e^{x\ln\frac{x+a}{x-a}} = \lim_{x\to+\infty} e^{x\ln\left(1+\frac{2a}{x-a}\right)} \overset{t=\frac{1}{x}}{=\!=} \lim_{t\to0^+} e^{\frac{\ln\left(1+\overset{\nearrow 0}{\boxed{\frac{2at}{1-at}}}\right)}{t}}$$

$$=\lim_{x\to0^+} e^{\underbrace{\boxed{\frac{\ln\left(1+\frac{2at}{1-at}\right)}{\frac{2at}{1-at}}}}_{\to1}\cdot\frac{2at}{t(1-at)}} = \lim_{t\to0^+} e^{\frac{2a}{1-at}} = e^{2a}$$

II način:

$$=\lim_{x\to+\infty}\left(1+\frac{2a}{x-a}\right)^x = \lim_{x\to\infty}\boxed{\left(1+\frac{1}{\frac{x-a}{2a}}\right)^{x\cdot\boxed{\frac{x-a}{2a}}}}^{\frac{2a}{x-a}} = \lim_{x\to+\infty}\left(1+\frac{1}{\frac{x-a}{2a}}\right)^{\frac{x-a}{2a}\cdot\frac{2ax}{x-a}}$$

$$=\lim_{x\to+\infty} e^{\overset{\to 1}{\boxed{\frac{2ax}{x-a}}}} = e^{2a}$$

$$14.\ \lim_{x\to-\infty}\left(\frac{3x^2-x+1}{3x^2+x+1}\right)^{\frac{x^2+1}{x-1}} = \lim_{x\to-\infty}\left(1-\frac{2x}{3x^2+x+1}\right)^{\frac{x^2+1}{x-1}} = \lim_{x\to-\infty}\underbrace{\left(1-\frac{1}{\underbrace{\boxed{\frac{3x^2+x+1}{2x}}}_{\to-\infty}}\right)^{\boxed{\frac{3x^2+x+1}{2x}}}}_{\to e}{}^{\cdot\boxed{-\frac{(x^2+1)}{x-1}}\cdot\frac{2x}{(3x^2+x+1)}}$$

$$=\lim_{x\to-\infty} e^{-\frac{(x^2+1)2x}{(x-1)(3x^2+x+1)}} = \lim_{x\to-\infty} e^{-\frac{(x^2+1)}{(3x^2+x+1)}\cdot\frac{2x}{x-1}} = \lim_{x\to-\infty} e^{-\frac{2}{3}}$$

$$15.\ \lim_{x\to-\infty}\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+b^2}+x} \overset{t=-x}{=\!=} \lim_{t\to+\infty}\frac{\sqrt{t^2+a^2}-t}{\sqrt{t^2+b^2}-t} = \lim_{t\to+\infty}\frac{\cancel{t}\left(\sqrt{1+\frac{a^2}{t^2}}-1\right)}{\cancel{t}\left(\sqrt{1+\frac{b^2}{t^2}}-1\right)}$$

$$\overset{t=\frac{1}{u}}{=\!=} \lim_{u\to0^+}\frac{(1+a^2u^2)^{1/2}-1}{(1+b^2u^2)^{\frac{1}{2}}-1} = \lim_{u\to0^+}\frac{\frac{((1+a^2u^2)^{1/2}-1)\cdot a^2u^2}{a^2u^2}}{\frac{((1+b^2u^2)^{1/2}-1)\cdot b^2u^2}{b^2u^2}} = \lim_{u\to0^+}\frac{\frac{a^2\cancel{u^2}}{\cancel{2}}}{\frac{b^2\cancel{u^2}}{\cancel{2}}} = \frac{a^2}{b^2}$$

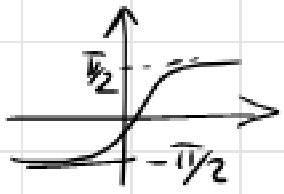
$$16.\ \lim_{x\to\pi}\frac{\sqrt{1-tg x}-\sqrt{1+tg x}}{\sin 2x} \overset{x=\pi+t}{=} \lim_{t\to0}\frac{\sqrt{1-tg t}-\sqrt{1+tg t}}{\sin 2t}\cdot\frac{\sqrt{1-tg t}+\sqrt{1+tg t}}{\sqrt{1-tg t}+\sqrt{1+tg t}} = \lim_{t\to0}\frac{-\cancel{2}\,tg\,t^{=\frac{\sin t}{\cos t}}}{\cancel{2}\sin t\cos t\cdot\left(\sqrt{1-tg t}+\sqrt{1+tg t}\right)}$$

$tg(\pi+t) = \frac{\sin(\pi+t)}{\cos(\pi+t)} = \frac{-\sin t}{-\cos t} = tg\,t$

$\sin(2\pi+2t) = \sin 2t$

$$=\lim_{t\to0}\frac{-\frac{1}{\boxed{\cos t}}\to 1}{\underset{\downarrow 1}{\boxed{\cos t}}\left(\underset{\searrow 1}{\sqrt{1-tg t}}+\underset{\downarrow 1}{\sqrt{1+tg t}}\right)} = \frac{-1}{2}$$

$$17.\ \lim_{x\to1^+}\operatorname{arctg}\frac{1}{1-x} \overset{x=1+t}{=} \lim_{x\to0^+}\operatorname{arctg}\frac{1}{-t} \overset{-\frac{1}{t}=u}{=\!=} \lim_{u\to-\infty}\operatorname{arctg}u = -\pi/2$$



$$\lim_{x\to1^-}\operatorname{arctg}\frac{1}{1-x} \overset{x=1+t}{=\!=} \lim_{x\to0^-}\operatorname{arctg}\frac{1}{-t} \overset{-1/t=u}{=\!=} \lim_{u\to+\infty}\operatorname{arctg}u = \pi/2$$

Za domaći

$$\lim_{x\to\substack{0^+\\0^-}} \frac{1}{1+e^{\frac{1}{x}}}$$

19. $\lim\limits_{x\to+\infty} \frac{2x+7}{x-3} = 2 \leftarrow$ Dokazati po definiciji

$\varepsilon>0$ proizvoljno

$M>0$ proizvoljno tkd Ako je $x>M$ onda je $\left|\frac{2x+7}{x-3}-2\right|<\varepsilon$

$\Leftrightarrow \left|\frac{2x+7-2x+6}{|x-3|}\right| = \frac{13}{|x-3|} < \varepsilon \Leftrightarrow |x-3| > \frac{13}{\varepsilon}$

$M = \frac{13}{\varepsilon}+3$, $x>M$ $\quad x > \frac{13}{\varepsilon}+3$

$\underbrace{x-3}_{>0} > \frac{13}{\varepsilon} \Rightarrow |x-3| > \frac{13}{\varepsilon}$

⑳ $\lim\limits_{x\to0}\frac{1-\sqrt[3]{1+x}}{\ln(1-x)}$ ㉑ $\lim\limits_{x\to0}\frac{\ln(\cos\alpha x)}{\ln(\cos\beta x)}$ ㉒ $\lim\limits_{x\to1} x^{\frac{1}{1-x}}$

㉓ $\lim\limits_{x\to0}\frac{\sqrt{\cos x}-\sqrt[3]{\cos x}}{\sin^2 x}$ ㉔ $\lim\limits_{x\to4}\frac{\sqrt{1+2x}-3}{\sqrt{x}-2}$

Četvrti čas

Asimptotske oznake o i $\sim$

Imamo f i q funkcije

$q = o(f)$ $x\to a \Leftrightarrow \exists$ okolina U tačke a i beskonačno mala funkcija α u $\overset{o}{U} = U\setminus\{a\}$ $\left(\lim\limits_{x\to a}\alpha(x)=0\right)$ tdj. $f(x) = \alpha(x)\cdot g(x)\ \forall x\in\overset{o}{U}$

$f(x) = x-1$

$q(x) = (x^2-1)(x-1)$

$q(x) = o(f(x))$, $x\to1$

$q(x) = \underbrace{(x^2-1)}_{=\alpha(x)}f(x) \longrightarrow \lim\limits_{x\to1}(x^2-1)=0$

$q(x)$ $1-\cos x = o(\overset{f(x)}{x})$ $x\to0$

$\alpha(x) = \frac{1-\cos x}{x}$

$\lim\limits_{x\to0}\left(\frac{1-\cos x}{x}\right)\cdot\frac{x}{x} = 0$

$q(x) = o(1)$, $x\to a$

$\lim\limits_{x\to a} q(x) = 0$

OSOBINE:

1. $o(f)+o(f) = o(f)$, $x\to a$ 2. $o(o(f)) = o(f)$, $x\to a$ $\left(\begin{matrix}\text{NPR.}\\ q=o(o(f))\Rightarrow q=o(f)\end{matrix}\right)$

3. $f\cdot o(q) = o(f\cdot q)$ $x\to a$ $\left(\begin{matrix}\text{NPR}\\ \frac{1}{x^2}\cdot o(x^2)=o(1)\end{matrix}\right)$

$h = f\cdot o(q)$, $x\to a$; $h(x) = f(x)\,\alpha(x)\,q(x)$; $\lim\limits_{x\to a}\alpha(x) = 0$; $h(x) = \alpha(x)(f\cdot q)(x)$

$h = o(f\cdot q)$ $x\to a$

4. $o(f)+o(o(f))=o(f),\ x\to a$

5. $c\in\mathbb{R}\setminus\{0\}$ $\quad$ (Npr: $o(2x)=o(x)$)

$c\cdot o(f)=o(c\cdot f)=o(f),\ x\to a$

6. $o(c+o(f))=o(f),\ x\to a$

Osnovni:

1. $\lim\limits_{x\to 0}\frac{\sin x}{x}=1$

$\lim\limits_{x\to 0}\frac{\sin x-x}{x}=0 \Rightarrow q(x)=\sin x-x,\ f(x)=x \Rightarrow \sin x-x=o(x),\ x\to 0$

$\sin x=x+o(x),\ x\to 0$

2. $\lim\limits_{x\to 0}\frac{1-\cos x}{x^2}=1/2\ ;\ \lim\frac{1-\cos x-\frac{1}{2}x^2}{x^2}=0 \Rightarrow 1-\cos x-\frac{1}{2}x^2=o(x^2)\ x\to 0$

$\Rightarrow \cos x=1-\frac{1}{2}x^2+o(x^2),\ x\to 0$

3. $\lim\limits_{x\to 0}\frac{e^x-1}{x}=1\quad \lim\limits_{x\to 0}\frac{e^x-1-x}{x}=0 \Rightarrow e^x-1-x=o(x),\ x\to 0 \Rightarrow e^x=1+x+o(x),\ x\to 0$

4. $\lim\limits_{x\to 0}\frac{\ln(1+x)}{x}=1 \Rightarrow \ln(1+x)=x+o(x),\ x\to 0$

5. $\lim\limits_{x\to 0}\frac{(1+x)^\alpha-1}{x}=\alpha \Rightarrow (1+x)^\alpha=1+\alpha x+o(x),\ x\to 0$ $\quad$ * $\ln(1+t)=t+o(t),\ t\to 0$

Primer: $\lim\limits_{x\to 0}\frac{\ln(\cos\alpha x)}{\ln(\cos\beta x)}=\lim\limits_{x\to 0}\frac{\ln(1-\overbrace{(\frac{1}{2}\alpha^2x^2+o(x^2))}^{t})}{\ln(1-\frac{1}{2}\beta^2x^2+o(x^2))}=\lim\limits_{x\to 0}\frac{-\frac{1}{2}\alpha^2x^2+o(x^2)+o(-\frac{1}{2}\alpha^2x^2+o(x^2))}{-\frac{1}{2}\beta^2x^2+o(x^2)+o(-\frac{1}{2}\beta^2x^2+o(x^2))}=$

$=\lim\limits_{x\to 0}\frac{-\frac{1}{2}\alpha^2x^2+o(x^2)}{-\frac{1}{2}\beta^2x^2+o(x^2)}=\lim\limits_{x\to 0}\frac{\alpha^2+\frac{o(x^2)}{x^2}}{\beta^2+\frac{o(x^2)}{x^2}}=\lim\limits_{x\to 0}\frac{\alpha^2+\overset{\to 0}{o(1)}}{\beta^2+\underset{\to 0}{o(1)}}=\frac{\alpha^2}{\beta^2}$

① $a,b,c>0\quad a,b,c\neq 1$

$\lim\limits_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{1/x}=\sqrt[3]{abc}\qquad \lim\limits_{x\to 0}e^{1/x\ln\left(\frac{a^x+b^x+c^x}{3}\right)}=\lim\limits_{x\to 0}e^{\frac{1}{x}\left(1+\overbrace{\frac{(a^x-1)+(b^x-1)+(c^x-1)}{3}}^{t}\right)}$

$=\lim\limits_{x\to 0}e^{\frac{1}{3x}((a^x-1)+(b^x-1)+(c^x-1)+o(a^x+b^x+c^x-3))}=\lim\limits_{x\to 0}e^{\frac{1}{3}\cdot\left(\frac{a^x-1}{x}+\frac{b^x-1}{x}+\frac{c^x-1}{x}\right)+o\left(\frac{a^x+b^x+c^x-3}{3\cdot x}\right)}$

$=\lim e^{\frac{1}{3}(\ln a+\ln b+\ln c)}\cdot e^{o\left(\frac{a^x+b^x+c^x-3}{3x}\right)}$ $\quad$ * $\lim\limits_{x\to 0}\frac{a^x-1}{x}=\lim\limits_{x\to 0}\frac{e^{x\ln a}-1}{x}=\lim\limits_{x\to 0}\frac{e^{x\ln a}-1}{x\ln a}\cdot\ln a=\ln a$

$=e^{\ln(abc)^{1/3}}\cdot\lim\limits_{x\to 0}e^{o\left(\frac{a^x+b^x+c^x-3}{3x}\right)}$ $\quad\rightarrow \ln\sqrt[3]{abc}=\ln\sqrt[3]{abc}+o(1)$

$=\sqrt[3]{abc}\cdot\lim\limits_{x\to 0}e^{o(\ln\sqrt[3]{abc}+o(1))}=\sqrt[3]{abc}$

② $L = \lim\limits_{x\to 0} \frac{1-\cos x^{\sin x}}{x^3}$

$\cos x^{\sin x} = e^{\sin x \ln \cos x} = e^{(x+o(x))\ln(1-\overset{t=}{\boxed{\frac{x^2}{2}+o(x^2)}})} = e^{(x+o(x))(-\frac{x^2}{2}+o(x^2))} =$

$= e^{-\frac{x^3}{2}+x\,o(x^2)-\frac{x^2}{2}o(x)+o(x)o(x^2)} = e^{-\frac{x^3}{2}+o(x^3)} = 1+\left(-\frac{x^2}{2}+o(x^3)\right)+o\left(-\frac{x^2}{2}+o(x^3)\right) =$

$= 1-\frac{x^2}{2}+o(x^3),\ x\to 0$

$L = \lim\limits_{x\to 0}\frac{1-(1-\frac{x^2}{2}+o(x^3))}{x^3} = \lim\limits_{x\to 0}\frac{\frac{x^2}{2}+o(x^3)}{x^3} = \lim\limits_{x\to 0}\frac{1/2+\overset{\to 0}{\boxed{o(1)}}}{1} = 1/2$

! $q\sim f,\ x\to a \quad \lim\limits_{x\to a}\frac{q(x)}{f(x)} = 1 \Rightarrow \lim\limits_{x\to a}\alpha(x) = 1$ isto što i: $q = o(f),\ x\to a \Rightarrow q(x) = \alpha(x)\cdot f(x)$

$\lim\limits_{x\to a}\alpha(x) = 0$ **!**

$\sin x\sim x,\ x\to 0 \mid e^x\sim 1+x,\ x\to 0 \mid \ln(1+x)\sim x,\ x\to 0$

$q\sim f,\ x\to a;\ q-f = o(f),\ x\to 0;\ q\sim f,\ x\to a;\ o(f) = o(q)\ x\to a$

3. $f(x)\quad c=?\ n=?\quad c\in\mathbb{R},\ n\in\mathbb{N}$

$f(x)\sim c\cdot x^n,\ x\to 0$

a) $f(x) = 2x+x^3-x^5 \quad f(x)\sim 2x,\ x\to 0$

$f(x) = 2x(1+\underset{\to 0}{\boxed{\frac{1}{2}\cdot x^2}}-\underset{\to 0}{\boxed{1/2\,x^4}})$

$\lim\limits_{x\to 0}\frac{f(x)}{2x} = 1 = \lim\limits_{x\to 0}\frac{2x(1+1/2\cdot x^2-1/2\cdot x^4)}{2x}$

b) $f(x) = \sqrt{1+x}-\sqrt{1-x}$

$\left.\begin{array}{l}\sqrt{1+x} = 1+1/2\,x+o(x),\ x\to 0\\ \sqrt{1-x} = 1-1/2\,x+o(x),\ x\to 0\end{array}\right\}\overset{\ominus}{\Rightarrow} f(x) = x+o(x),\ x\to 0$

$\sqrt{x+1}\sim 1+1/2\,x,\ x\to 0 \mid \sqrt{x-1}\sim 1-1/2\cdot x,\ x\to 0$

$f(x)\sim 1+x/2-1+x/2,\ x\to 0 \Rightarrow f(x)\sim x,\ x\to 0$

4. $\lim \frac{(1+\cos x)^{1-\cos x}-1}{(\ln(1+x))^2}$

$(1+\cos x)^{1-\cos x} = e^{(1-\cos x)\ln(1+\cos x)} = e^{(\frac{x^2}{2}+o(x^2))\cdot\ln(1+1-\frac{x^2}{2}+o(x^2))} =$

$= e^{(\frac{x^2}{2}+o(x^2))\ln(2-\frac{x^2}{2}+o(x^2))} = e^{(\frac{x^2}{2}+o(x^2))(\ln 2+\overset{=t}{\boxed{\ln(1-\frac{x^2}{4}+o(x^2))}})} =$

$= e^{(\frac{x^2}{2}+o(x^2))(\ln 2-\frac{x^2}{4}+o(x^2)+o(-\frac{x^2}{4}+o(x^2))} = e^{(\frac{x^2}{2}+o(x^2))(\ln 2-\frac{x^2}{4}+o(x^2))} =$

$= e^{(\frac{x^2}{2}\ln 2-\underbrace{\frac{x^4}{8}+\frac{x^2}{2}\cdot o(x^2)+o(x^2)\ln 2-\frac{x^2}{4}o(x^2)+o(x^2)o(x^2)}_{\to o(x^2)})} = e^{\overset{=t}{\boxed{\frac{x^2}{2}\ln 2+o(x^2)}}}$

$\downarrow$

$$= 1 + \frac{x^2}{2}\ln 2 + o(x^2) + o\left(\frac{x^2}{2}\ln 2 + o(x^2)\right) = 1 + \frac{x^2}{2}\ln 2 + o(x^2)$$

$$[\ln(1+x))^x = (x + o(x))^2 = x^2 + 2o(x)\cdot x + o(x)\cdot o(x) = x^2 + 2o(x^2) + o(x^2) = x^2 + o(x^2), x \to 0$$

$$\lim_{x\to 0} \frac{1 + \frac{x^2}{2}\ln 2 + o(x^2) - 1}{x^2 + o(x^2)} = \lim_{x\to 0} \frac{\frac{\ln 2}{2} + o(1)}{1 + o(1)} = \frac{\ln 2}{2}$$

5. $\lim\limits_{x\to 0} \frac{\cos 3x + e^{x^2} - 2}{\ln(1+5x)\sin 2x}$; $\lim\limits_{x\to 0} \frac{1 - \frac{(3x)^2}{2} + o((3x)^2) + 1 + x^2 + o(x^2) - 2}{(5x + o(5x))(2x + o(2x))} = \lim\limits_{x\to 0} \frac{-\frac{9x^2}{2} + x^2 + o(x^2)}{(5x + o(5x))(2x + o(x))} =$

$$= \lim_{x\to 0} \frac{-\frac{7x^2}{2} + o(x^2)}{10x^2 + 5x\,o(x) + 2x\,o(x) + o(x)o(x)} = \lim_{x\to 0} \frac{-\frac{7}{2}\cdot\cancel{x^2} + o(\cancel{x^2})}{\cancel{x^2}(5 + o(1))(2 + o(1))} = \lim_{x\to 0} \frac{-\frac{7}{2} + o(1)}{\underbrace{(5 + o(1))}_{5}\,\underbrace{(2 + o(1))}_{2}}$$

$$= \frac{-\frac{7}{2}}{10} = -\frac{7}{20}$$

Peti čas

$f: A \to \mathbb{R}$, $a \in A$

f je neprekidna u tački a ako je $\lim\limits_{x\to a} f(x) = f(a)$

$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in D_f)\ |x - a| < \delta \Longrightarrow |f(x) - f(a)| < \varepsilon$

kažemo da je f neprekidna na A ako je neprekidna u a $\forall a \in A$.

f, g neprekidne $\Rightarrow f \pm g, f\cdot g$ neprekidne

$\forall x \in A\ g(x) \neq 0 \Rightarrow \frac{f}{g}$ neprekidna

1. $f: \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

Da li je neprekidna?

$x \neq 0$:

$\left.\begin{array}{l} \sin x \text{ neprekidna} \\ x \text{ neprekidna} \end{array}\right\} \Rightarrow \frac{\sin x}{x}$ je neprekidna u $x \in \mathbb{R}\setminus\{0\}$

Da li je neprekidna u 0?

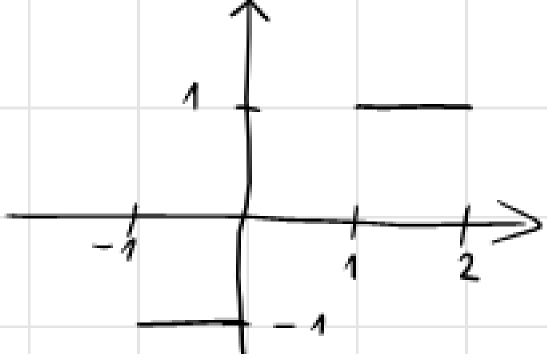
$\lim\limits_{x\to 0} f(x) = f(0) \Leftrightarrow \lim\limits_{x\to 0} \frac{\sin x}{x} = 1$ Ⓣ f je neprekidna

2. $f: \mathbb{R} \to \mathbb{R}$ $f(x) = \begin{cases} \frac{\sin x}{|x|}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

$\left.\begin{array}{l} \lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} \frac{\sin x}{-x} = -1 \\ \lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} \frac{\sin x}{x} = 1 \end{array}\right\}$ $\lim\limits_{x\to 0} f(x)$ ne postoji $\Rightarrow$ f-ja nije neprekidna u 0.

3. $f: \mathbb{R} \to \mathbb{R}$

$f(x) = [X]$



$$\left.\begin{array}{l}\lim\limits_{x \to k^-} f(x) = k-1 \\ \lim\limits_{x \to k^+} f(x) = k\end{array}\right\}$$ k nije neprekidna u $\mathbb{Z}$ ali jeste u $\mathbb{R} \setminus \mathbb{Z}$

4. $f: \mathbb{R} \to \mathbb{R}$

$f(x) = [x]\left([x] - (-1)^{[x]} \cdot \cos \pi x\right)$

a) $x = k,\ k \in \mathbb{Z}$

$f(x) = k(k - (-1)^k \cos \pi k) = k(k - (-1)^k \cdot (-1)^k) = k(k-1) = f(k)$

b) $x \in (k, k+1),\ k \in \mathbb{Z}$

$f(x) = \underbrace{k\ (k}_{\text{konstante}} - \underbrace{(-1)^k \cos \pi x}_{\text{neprekidna}})$

$f(x)$ neprekidna u $x \Leftarrow \left\{\begin{array}{l}[x] \text{ neprekidna u } x \text{ za } x \in \mathbb{R} \setminus \mathbb{Z} \\ \cos \pi x \text{ neprekidna}\end{array}\right.$

Vratimo se u a) $k \in \mathbb{Z}$

1 $\lim\limits_{x \to k^-} f(x) = \lim\limits_{x \to k^-} (k-1)((k-1) - (-1)^{k-1} \cos \pi x) = (k-1)((k-1) - (-1)^{k-1} \cdot (-1)^k) = (k-1)(k-1-(-1)) =$
$= (k-1) \cdot k$

2. $\lim\limits_{x \to k^+} f(x) = \lim\limits_{x \to k^+} k(k - (-1)^k \cos \pi x) = k(k - (-1)^k (-1)^k) = k(k-1)$

$\lim\limits_{x \to k^-} f(x) = \lim\limits_{x \to k^+} f(x) = k(k-1) \Rightarrow \lim\limits_{x \to k} f(x) = k(k-1)$

$f(k) = k(k-1) \Rightarrow \lim\limits_{x \to k} f(x) = f(k) \Longrightarrow f$ je neprekidna u $k \Rightarrow f$ je neprekidna

5. $f: \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} \dfrac{1}{1 + e^{\frac{1}{x-1}}}, & x \neq 1 \\ 1, & x = 1 \end{cases}$$

$\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} \dfrac{1}{1 + e^{\frac{1}{x-1}}} = \left\lceil \begin{array}{l}\text{SMENA} \\ 1 - t = x\end{array} \right\rfloor = \lim\limits_{t \to 0^-} \dfrac{1}{1 + e^{-1/t}} = \left\lceil \begin{array}{l}\text{SMENA} \\ u = -\frac{1}{t}\end{array} \right\rfloor = \lim\limits_{u \to -\infty} \dfrac{1}{1 + \underbrace{\boxed{e^u}}_{\to 0}} = 1$

$\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} \dfrac{1}{1 + e^{\frac{1}{x-1}}} = \left\lceil \begin{array}{l}\text{SMENA} \\ 1 + t = x\end{array} \right\rfloor = \lim\limits_{t \to 0^+} \dfrac{1}{1 + e^{+1/t}} = \left\lceil \begin{array}{l}\text{SMENA} \\ u = +\frac{1}{t}\end{array} \right\rfloor = \lim\limits_{u \to +\infty} \dfrac{1}{1 + \underbrace{\boxed{e^u}}_{\infty}} = 0$

$\Rightarrow f$ nije neprekidna u $x = 1$

6. $f:(-1,+\infty)\setminus\{0\} \to \mathbb{R}$

$f(x)=\dfrac{\sqrt{x+1}-1}{\sqrt[3]{x+1}-1}$ Da li možemo da dodefinišemo u 0 tkd. bude neprekidna u 0?

$g:(-1,+\infty)\to\mathbb{R}$

$g(x)=\begin{cases} f(x), & x\neq 0\\ a, & x=0\end{cases}$ Možemo li da nađemo ovakvu funkciju da bude neprekidna?

ako $\exists g$ onda $a=\lim\limits_{x\to 0} f(x)$

$$\lim_{x\to 0} f(x)=\lim_{x\to 0}\frac{\sqrt{x+1}-1}{\sqrt[3]{x+1}-1}\cdot\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1}=\lim_{x\to 0}\frac{x}{(\sqrt[3]{x+1}-1)(\sqrt{x+1}+1)}\cdot\frac{\sqrt[3]{x+1}+\sqrt[3]{(x+1)^2}+1}{\sqrt[3]{x+1}+\sqrt[3]{(x+1)^2}+1}=$$

$$=\lim_{x\to 0}\frac{\cancel{x}(\sqrt[3]{x+1}+\sqrt[3]{(x+1)^2}+1)}{\cancel{x}(\sqrt{x+1}+1)}=\frac{3}{2}$$

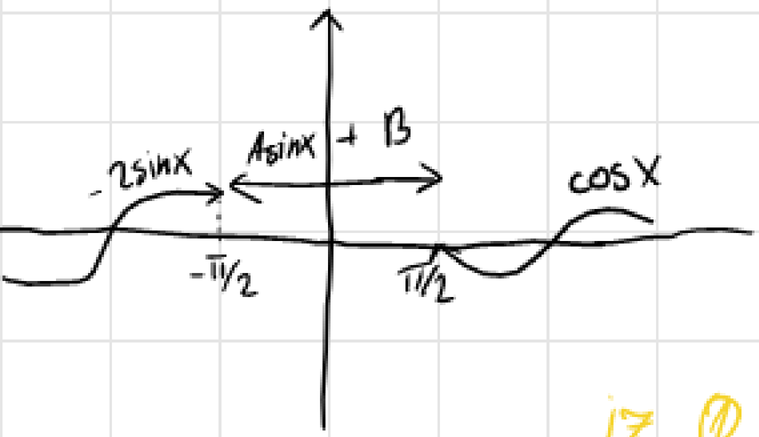
Drugi način: $$\lim_{x\to 0}\frac{\sqrt{x+1}-1}{\sqrt[3]{x+1}-1}=\lim_{x\to 0}\frac{\overbrace{\boxed{\frac{\sqrt{1+x}-1}{x}}}^{\to 1/2}\cdot\cancel{x}}{\underbrace{\boxed{\frac{\sqrt[3]{1+x}-1}{x}}}_{\to 1/3}\cdot\cancel{x}}=\frac{3}{2}$$

$g=\begin{cases} f(x), & x\neq 0\\ 3/2, & x=0\end{cases} \Longrightarrow \lim\limits_{x\to 0} g(x)=g(0)\Rightarrow g$ je neprekidna tj. dodefinisali smo f

! Tačke u kojima funkcija nije nedefinisana nas ne zanimaju kad gledamo neprekidnost !

7. $f(x)=\begin{cases} -2\sin x, & x\leq -\pi/2\\ A\sin x+B, & |x|<\frac{\pi}{2}\\ \cos x, & x\geq \pi/2\end{cases}$ Da li postoje $A,B\in\mathbb{R}$ tkdj. f neprekidna?

$$\left.\begin{aligned}&\lim_{x\to -\pi/2^-} f(x)=\lim_{x\to -\pi/2^-}(-2\sin x)=-2(-1)=2\\ &\lim_{x\to -\pi/2^+} f(x)=\lim_{x\to -\pi/2^+} A\sin x+B=-A+B\end{aligned}\right\}$$ Da bi f bilo neprekidno u $-\pi/2$ mora da važi $-A+B=2$ tj $B=A+2$ ①



$$\left.\begin{aligned}&\lim_{x\to \pi/2^+} f(x)=\lim_{x\to\pi/2^+}\cos x=0\\ &\lim_{x\to \pi/2^-} f(x)=\lim_{x\to\pi/2^-}(A\sin x+B)=A+B\end{aligned}\right\}$$ $A+B=0$
$A=-B$ ②

iz ① i ② $A=-1$ $B=1$

↓Nastavak ↓

8. Da li su tačni iskazi?

a) f je neprekidna u x_0, q ima prekid u $x_0 \Rightarrow f+q$ ima prekid u x_0?

$(\forall \varepsilon>0)(\exists \delta>0)(\forall x \in D_f)|x-x_0|<\delta \Rightarrow |f(x)-f(x_0)|<\varepsilon$ za f

$(\exists \varepsilon>0)(\forall \delta>0)(\exists x \in D_f)|x-x_0|<\delta \Rightarrow |q(x)-q(x_0)| \geq \varepsilon_0$ za q $\quad D_f = R$

<u>ili</u>:

$\exists \lim\limits_{x \to x_0} f(x) = f(x_0)$

PPS: $f+q$ neprekidna u $x_0 \Rightarrow \lim\limits_{x\to x_0}(f(x)+q(x)) = f(x_0)+q(x_0)$

$\Rightarrow \lim\limits_{x\to x_0}(f(x)+q(x)-f(x)) = f(x_0)-f(x_0)+q(x_0) = q(x_0)$

$\Rightarrow \lim\limits_{x\to x_0} q(x) = q(x_0)$ ↯ q nije neprekidna u x_0

Vraćamo se na I način:

$\varepsilon_0>0$ tdj. $(\exists \delta_1>0)(\exists x \in R)|x-x_0|<\delta_1 \wedge |q(x)-q(x_0)| \geq \varepsilon$

f neprekidna, uzmemo $\varepsilon = \frac{\varepsilon_0}{2}$ $(\exists \delta_2>0)(\forall x \in R)|x-x_0|<\delta_2 \Rightarrow |f(x)-f(x_0)| < \frac{\varepsilon}{2}$

Uzmemo $\delta_1 = \delta_2$ (min) $\Rightarrow (\exists x \in R)|x-x_0|<\delta_1 \wedge \underbrace{|q(x)-q(x_0)|}_{G} \geq \varepsilon_0$

$|x-x_0|<\delta_2\ (=\delta_1) \Rightarrow \underbrace{|f(x)-f(x_0)|}_{F} < \frac{\varepsilon_0}{2}$

$|G| \geq \varepsilon_0 \quad |F| < \frac{\varepsilon_0}{2} \quad |G+F| = ?$

$|G+F| \geq |G|-|F| > \varepsilon_0 - \frac{\varepsilon_0}{2} = \frac{\varepsilon_0}{2} \Rightarrow |f(x)+q(x)-f(x_0)-q(x_0)| > \frac{\varepsilon_0}{2}$

↳ nejednakost trougla

Pokazali smo za $\frac{\varepsilon_0}{2}>0$ važi: $(\forall \delta_1>0)(\exists x \in R)|x-x_0|<\delta_1 \wedge |f(x)+q(x)-f(x_0)-q(x_0)| > \frac{\varepsilon_0}{2}$

$\delta_1 \leq \delta_2$

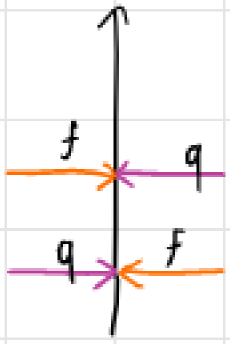
$\Rightarrow f+q$ ima prekid u x_0

b) f i q imaju prekid u $x_0 \Rightarrow f+q$ ima prekid u x_0

Videli smo $[x]([x]-(-1)^{[x]} \cdot \cos \pi x) = \overbrace{\underline{[x]^2} + \underline{(-[x](-1)^{[x]}\cos \pi x)}} \Rightarrow$ neprekidna u 0

ima prekid u 0

$\Rightarrow$ Ne mora da važi



c) f i q imaju prekid u $x_0 \Rightarrow f+q$ neprekidna u x_0; Ne mora jer $\overset{\text{ima}}{[x]} + \overset{\text{ima}}{[x]} = 2\overset{\text{ima}}{[x]}$

9. $f(x) = \text{sgn}(x) = \begin{cases} 1, & x>0 \\ 0, & x=0 \\ -1, & x<0 \end{cases}$ $\quad q(x) = 1+x-[x]$ Ispitati neprekidnost f-ja $f \circ q$ i $q \circ f$

! $f: A \to R \quad q \circ f: A \to R \quad (f(A) \subseteq B)$

$q: B \to R \quad f \circ q: B \to R \quad (q(B) \subseteq A)$

f i q neprekidne $\Rightarrow f \circ q$ i $q \circ f$ neprekidne !

$g\circ f:\mathbb{R}\to\mathbb{R}$

$g(f(x)) = g(\operatorname{sgn}x) = 1+\operatorname{sgn}x-[\operatorname{sgn}x] = 1+\operatorname{sgn}x-\operatorname{sgn}x = 1$

$\Rightarrow g\circ f:\mathbb{R}\to\mathbb{R}$ $(g\circ f)(x)=1$ - konst. $\Rightarrow g\circ f$ je neprekidna

$f\circ g:\mathbb{R}\to\mathbb{R}$

$f(g(x)) = \operatorname{sgn}(1+x-[x])$ $\quad [x]\le x \Rightarrow [x]<x+1 \Rightarrow x+1-[x]>0$ smena $y>0$

$\Rightarrow (f\circ g)(y)=1 \Rightarrow f\circ g$ neprekidna

⑩ $f:(-\infty,1]\to\mathbb{R}$ $\quad f(x)=\begin{cases}\dfrac{\cos 2x+e^{x^2}-2}{(\sqrt{1+3x}-1)\ln(1+x)}, & x\in(0,1]\\ \sqrt[3]{x+1}\cdot|x^2+x|+a, & x\in(-\infty,0]\end{cases}$ $\quad a=?$ tdj. f neprekidna

$\lim\limits_{x\to0^-} f(x) = \lim\limits_{x\to0^-}(\overset{\nearrow 1}{\sqrt[3]{x+1}}\cdot\overset{\nearrow 0}{|x^2+x|}+a) = a \qquad f(0)=\sqrt[3]{1}\cdot(0+0)+a=a$

$\lim\limits_{x\to0^+} f(x) = a?$ $\quad \lim\limits_{x\to0^+} f(x) = \lim\limits_{x\to0^+}\dfrac{\cos 2x+e^{x^2}-2}{(\sqrt{1+3x}-1)\ln(1+x)} = \lim\limits_{x\to0^+}\dfrac{\cancel{1}-\frac{(2x)^2}{2}+o(x^2)+\cancel{1}+x^2+o(x^2)-\cancel{2}}{(\frac{1}{2}\cdot3x+o(x))(x+o(x))} =$

$= \lim\limits_{x\to0^+}\dfrac{-2x^2+x^2+o(x^2)}{x^2(3/2+o(1))(1+o(1))} = \lim\limits_{x\to0^+}\dfrac{-1+o(1)}{\frac{3}{2}} = -\dfrac{2}{3}$ za $a=-\frac{2}{3}$ f je neprekidna

Drugi način:

$\lim\limits_{x\to0^+}\dfrac{\cos 2x+e^{x^2}-2}{(\sqrt{1+3x}-1)\ln(1+x)} = \lim\limits_{x\to0^+}\dfrac{\frac{\cos 2x-1}{(2x)^2}\cdot(2x)^2+\frac{e^{x^2}-1}{x^2}\cdot x^2}{\frac{\sqrt{1+3x}-1}{3x}\cdot3x\cdot\frac{\ln(1+x)}{x}\cdot x} = \lim\limits_{x\to0^+}\dfrac{-\frac{1}{2}\cdot4x^2+x^2}{\frac{3}{2}x^2} = -2/3$

⑪ $f:\mathbb{R}\setminus\{0\}\to\mathbb{R}$ $\quad f(x)=\begin{cases}ax+b, & x\in(-\infty,-1]\cup[1,+\infty)\\ \dfrac{|x|-1}{\ln|x|}, & x\in(-1,1)\setminus\{0\}\end{cases}$ $\quad a,b=?$ da je f neprekidna

$\lim\limits_{x\to1^+} f(x)=a+b \qquad f(1)=a+b$

$\lim\limits_{x\to1^-} f(x)=\lim\limits_{x\to1^-}\dfrac{|x|-1}{\ln|x|} = \left[\begin{matrix}\text{smena}\\ t=1-x\end{matrix}\right] = \lim\limits_{t\to0^-}\dfrac{-t}{\ln(1-t)} = \lim\limits_{t\to0^-}\dfrac{-t}{\frac{\ln(1-t)}{-t}\cdot(-t)} = 1 \Rightarrow \boxed{a+b=1}$

$\lim\limits_{x\to-1^+} f(x)=\lim\limits_{x\to-1^+}\dfrac{|x|-1}{\ln|x|} = \left[\begin{matrix}\text{smena}\\ t=x+1\end{matrix}\right] \lim\limits_{t\to0^+}\dfrac{|-1+t|-1}{\ln(1+t)} = \lim\limits_{t\to0^+}\dfrac{1-t-1}{\ln(1-t)} = 1$

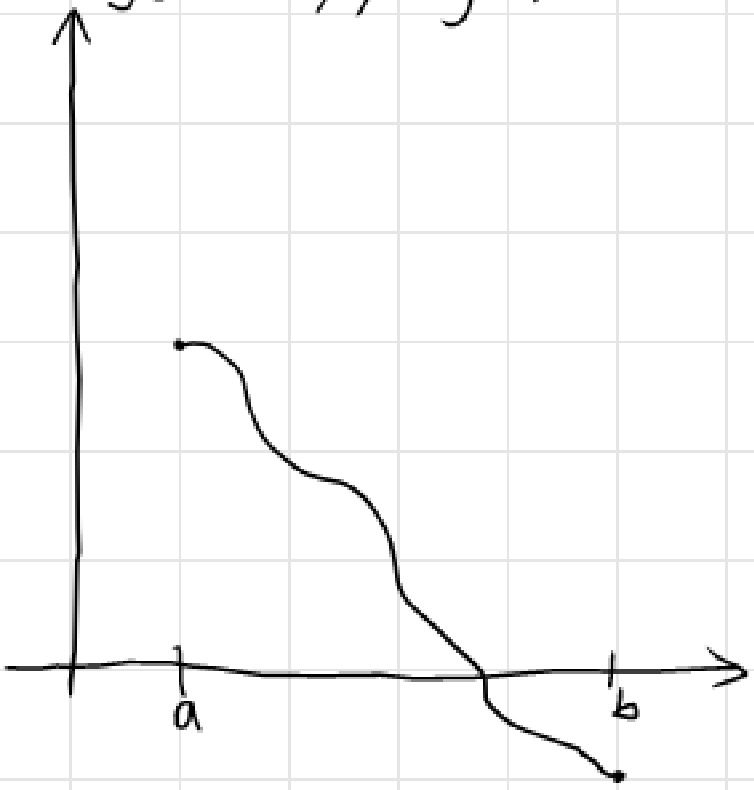
$\lim\limits_{x\to-1^-} f(x) = -a+b \qquad f(-1)=-a+b \qquad \Rightarrow \boxed{-a+b=1}$

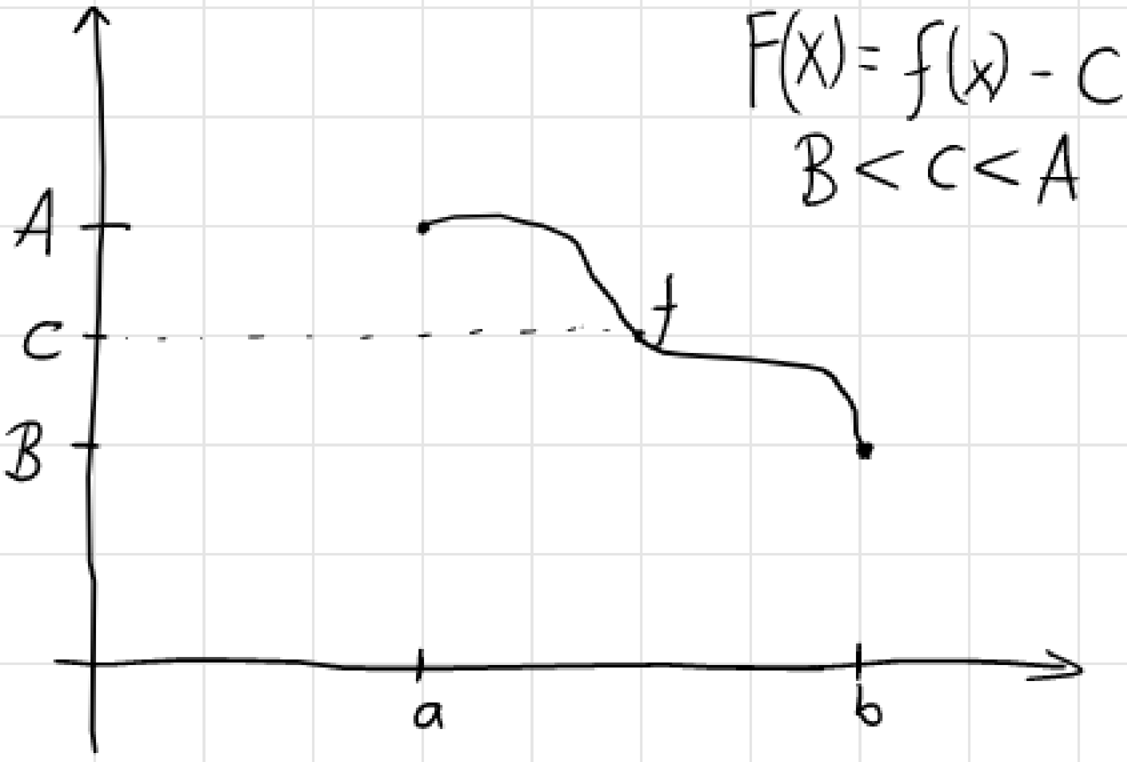
$b=1$
$a=0$

Šesti čas

Koši-Bolcanova: $f: [a,b] \to \mathbb{R}$ neprekidna $f(a)f(b)<0$

$\Rightarrow \exists c \in (a,b) \quad f(c)=0$





$F(x)=f(x)-C$

$B<C<A$

Teorema o meduvrednosti: $f\ [a,b] \to \mathbb{R}$ neprekidna, C vrednost izmedu $f(a)$ i $f(b)$

$\Rightarrow \exists c \in (a,b) \quad f(c)=C$

Vajerštras: $f[a,b] \to \mathbb{R}$ neprekidna $\Rightarrow f$ dostiže min i max

$(\exists \alpha \in [a,b])\ f(\alpha)=\min f(x),\ x\in[a,b]$

$(\exists \beta \in [a,b])\ f(\beta)=\max f(x),\ x\in[a,b]$

1. $f, q : [a,b] \to [a,b]$ q je „na" f-ja $\overset{?}{\Rightarrow} (\exists x_0 \in [a,b]\ f(x_0)=q(x_0))$

$f(x_0)=q(x_0) \Leftrightarrow f(x_0)-q(x_0)=0$

uzmemo $h(x)=f(x)-q(x)$, h neprekidna

$\Rightarrow h:[a,b] \to \mathbb{R}$

q je „na" $\Rightarrow \exists \alpha \in [a,b]\quad q(\alpha)=a$

$\exists \beta \in [a,b]\quad q(\beta)=b$

znamo ${}^{1}\ a \le f(\alpha) \le b$, ${}^{2}\ a \le f(\beta) \le b \Rightarrow {}^{1}\ f(\alpha)-q(\alpha) \ge a-a=0$, ${}^{2}\ f(\beta)-q(\beta) \le b-b=0$

$\Rightarrow h(\alpha) \ge 0 \wedge h(\beta) \le 0$ ako je $h(\alpha)=0 \to f(\alpha)=q(\alpha) \Rightarrow x_0=\alpha$

$h(\beta)=0 \to f(\beta)=q(\beta) \Rightarrow x_0=\beta$

$h(\beta) \ne 0 \wedge h(\alpha) \ne 0 \Rightarrow h(\beta)<0 \wedge h(\alpha)>0 \xrightarrow{K.B} \exists x_0$ izmedu α i β td. $h(x_0)=0$

$\Rightarrow f(x_0)=q(x_0)$

Da li postoje skupovi A i B td. $A\cap B=\emptyset$ i $A\cup B=[0,1]$ i f-ja $f:[0,1]\to[0,1]$ neprekidna $f(A)\subseteq f(B) \wedge f(B)\subseteq A$?

2. $f:\mathbb{R}\to\mathbb{R}$ neprekidna td. $(\exists c\in\mathbb{R})\ f(f(c))=c$. Dokazati da postoji $d\in\mathbb{R}$ td. $f(d)=d$.

Tražimo $\left.\begin{array}{l} f(a)>a \to f(a)-a>0 \\ f(b)<b \to f(b)-b>0 \end{array}\right\} \xRightarrow{KB} \exists x_0\ f(x_0)-x_0=0$ jer $\begin{array}{l} f(d)-d=0 \\ f(x)-x=0 \end{array}$

Posmatramo $f(c)$, c i definišemo novu funkciju $q(x)=f(x)-x$ $\quad q(c)=f(c)-c$

$q(f(c))=f(f(c))-f(c)=c-f(c) \Longleftrightarrow q(f(c))=-q(c)$

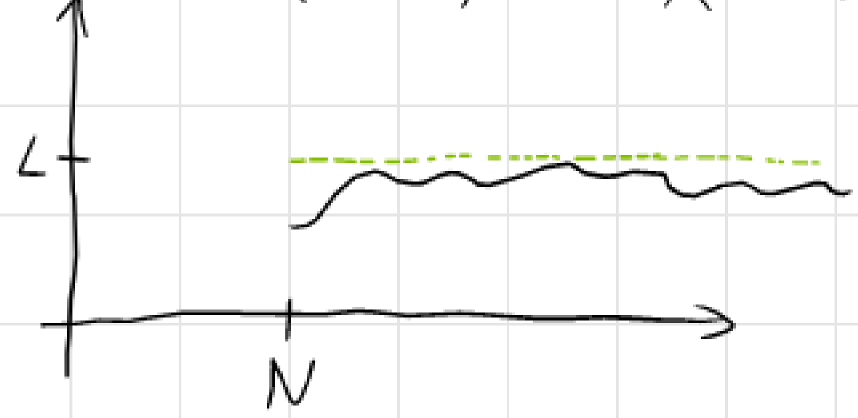
I slučaj $q(c)=0$ td. $f(c)=c \Rightarrow d=c$

II slučaj $q(c)\neq 0$, $f(c)\neq c$ $\quad q(c)\cdot q(f(c))=-(q(c))^2<0 \xRightarrow{K.B.} \exists d$ između $f(c)$ i c td. $q(d)=0$ tj $f(d)-d=0 \Rightarrow f(d)=d$

③ $f:[5,+\infty)\to\mathbb{R}$ neprekidna $\exists \lim\limits_{x\to+\infty} f(x)=L\in\mathbb{R}$. Dokazati da je f ograničena na $[5,+\infty)$

$(\exists M>0)(\forall x\in[5,+\infty))\ |f(x)|\leq M$

$\exists \lim$ znači $(\forall \varepsilon>0)(\exists N>0)(\forall x\in[5,+\infty))\ x>N \Rightarrow |f(x)-L|<\varepsilon$



$L-\varepsilon<f(x)<L+\varepsilon$

npr. za $\varepsilon=1$ $(\exists N>0)(\forall x>N)\ L-1<f(x)<L+1$

Posmatramo $[5,N]$, f neprekidna

$\xRightarrow{Vajer.}$ dostiže min i max $\quad \min f(x)=\alpha,\ x\in[5,N] \wedge \max f(x)=\beta,\ x\in[5,N]$

$x\in[5,N] \Rightarrow \alpha\leq f(x)\leq \beta$

$M=\max\{|\alpha|,|\beta|,|L+1|,|L-1|\} \Rightarrow (\forall x\in[5,+\infty))\ |f(x)|\leq M$

4. $7x^5-5x^3+3x-2=\sin\left(\frac{\pi}{2x}\right)$ Dokazati da jednačina ima rešenja na $(0,1)$

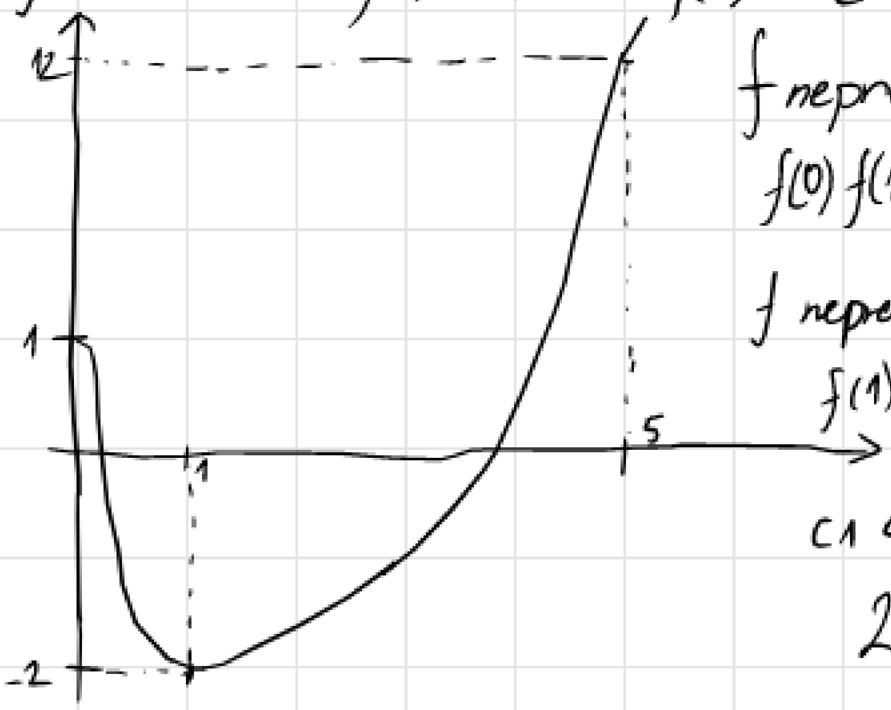
Neka je $f(x)=7x^5-5x^3+3x-2-\sin\left(\frac{\pi}{2x}\right)$ $\quad f(0)=?$, $f(1)=?$

$f(0)=-2<0$ $\quad f(1)=2>0 \Rightarrow f(0)f(1)<0$, f neprekidna $\xRightarrow{KB} (\exists c\in(0,1))\ f(c)=0$

↓Nastavak↓

5. Dokazati da $2^x = 4x$ ima bar 2 rešenja na $(0, +\infty)$

$f(x) = 2^x - 4x \quad f(0) = 1 > 0 \quad f(1) = -2 < 0 \quad f(5) = 12 > 0$



$\left.\begin{array}{l} f \text{ neprekidna na } [0,1] \\ f(0)f(1) < 0 \end{array}\right\} \xRightarrow{K.B.} \exists c_1 \in (0,1) \; f(c_1) = 0$

$\left.\begin{array}{l} f \text{ neprekidna na } [1,5] \\ f(1)f(5) < 0 \end{array}\right\} \xRightarrow{K.B.} \exists c_2 \in (1,5) \; f(c_2) = 0$

$c_1 < c_2 \Rightarrow c_1 \neq c_2 \quad f(c_1) = f(c_2) = 0 \Rightarrow$ Postoje bar 2 rešenja

6. $f : R \to R$ neprekidna i ograničena. Dokazati $\exists x_0 \in \mathbb{R} \; f(x_0) = x_0^3 \quad q(x) = x^3 - f(x)$

f ograničena $\Rightarrow (\exists M > 0)(\forall x \in \mathbb{R}) \; |f(x)| < M \quad -M < f(x) < M$

1 Biram x_1 tkd. $q(x_1) > 0 \iff x_1^3 - f(x_1) > 0 \iff x_1^3 > f(x_1)$ dovoljno da $x_1^3 \geq M$

$x_1 = \sqrt[3]{M} \Rightarrow q(x_1) = M - f(x_1) \xrightarrow{f(x_1) < M} M - f(x_1) > 0$

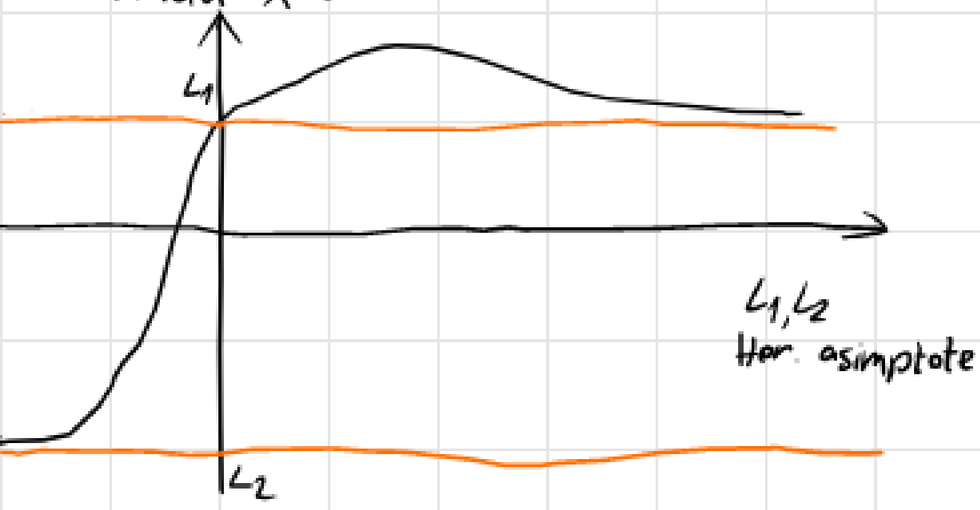
2 Biram x_2 tkd. $q(x_2) < 0 \iff x_2^3 - f(x_2) < 0 \iff x_2^3 \leq -M$

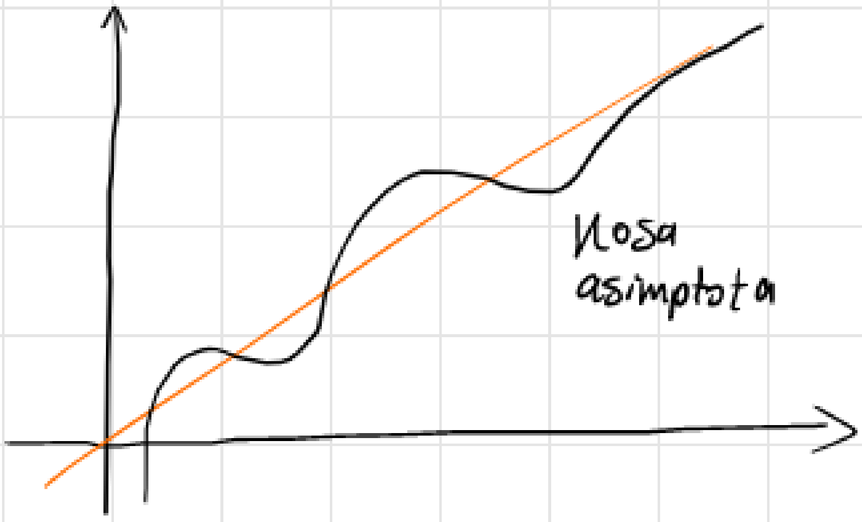
$x_2 = -\sqrt[3]{M} \Rightarrow q(x_2) = -M - f(x_2) \xrightarrow{f(x_2) > -M} -M - f(x_2) < 0$

Dobili smo $\left.\begin{array}{l} q(x_1) > 0 \\ q(x_2) < 0 \end{array}\right\} \xRightarrow{K.B.} \exists x_0$ između x_1 i x_2 $q(x_0) = 0$ tj. $f(x_0) = x_0^3$

7. Poligon u ravni (konveksan). Dokazati da postoji prava koja ga deli na 2 dela jednake površine i jednakog obima. (Neće biti na ispitu)

$f : \mathbb{R} \to \mathbb{R}$ ako $\exists \lim\limits_{x \to +\infty} f(x) = L \in \mathbb{R}$ onda f ima horizontalnu asimptotu $y = L$ kada $x \to +\infty$





Nastavak

$\lim_{x\to+\infty}(f(x)-(ax+b)) = 0 \Rightarrow f$ ima kosu asimptotu $y = ax+b,\ x\to+\infty$

Primeri:

1) $f(x) = \operatorname{arctg} x \quad \lim_{x\to+\infty} f(x) = \frac{\pi}{2} \quad \lim_{x\to-\infty} f(x) = -\pi/2$

ima horizontalnu asimptotu $y = \pi/2$ kada $x\to+\infty$ i $y = -\pi/2$ kada $x\to-\infty$

2) $f(x) = \sqrt{x^2+1} \quad \lim_{x\to+\infty} f(x) = +\infty \Rightarrow$ nema horizontalnu

$\lim_{x\to+\infty} f(x)\cdot\frac{1}{x} \rightarrow$ ako postoji jednako je a

$$\lim_{x\to+\infty}\frac{\sqrt{x^2+1}}{x} = \lim_{x\to+\infty}\sqrt{1+1/x^2} = 1 \Rightarrow \boxed{a=1} \quad \lim_{x\to+\infty}(f(x)-ax) = b \text{ (ako postoji)}$$

$$\lim_{x\to+\infty}\left(\sqrt{x^2+1}-x\right) = \begin{bmatrix}\text{Smena}\\ t=\frac{1}{x}\end{bmatrix} = \lim_{t\to0^+}\left(\sqrt{1+\frac{1}{t^2}}-\frac{1}{t}\right) = \lim_{t\to0^+}\left(t\cdot\frac{\sqrt{t^2+1}-1}{t^2}\right) = \lim_{t\to0^+}\frac{t}{2} = 0 \Rightarrow \boxed{b=0}$$

$\Rightarrow$ kosa asimptota $y = x$

3) $f(x) = \sqrt{x} \quad \lim_{x\to+\infty}\sqrt{x} = +\infty$ Nema $\quad \lim_{x\to+\infty}\frac{\sqrt{x}}{x} = 0$ Nema

⑧ $f(x) = \sqrt{\frac{x^3}{x-1}}$ Asimptote $x\to+\infty,\ x\to-\infty \quad f: (-\infty,0]\cup(1,+\infty)\to\mathbb{R}$

Horizontalne

- $+\infty$: $\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\sqrt{\frac{x^3}{x-1}} = \lim_{x\to+\infty}\sqrt{\frac{x^{\cancel{3}\,2}\cdot1}{\cancel{x}(1-\frac{1}{x})}} = +\infty$
- $-\infty$: $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\sqrt{\frac{x^3}{x-1}} = \lim_{x\to-\infty}\sqrt{\frac{\cancel{x}\cdot x^2\cdot1}{\cancel{x}(1-1/x)}} = +\infty$

} Nema horizontalne asimptote

Kose:

- $+\infty$: $\lim_{x\to+\infty}\frac{f(x)}{x} = \lim_{x\to+\infty}\sqrt{\frac{x^{\cancel{3}}}{\cancel{x^2}(x-1)}} = \lim_{x\to+\infty}\sqrt{\frac{x}{x-1}} = \boxed{1 = a_1}$

$$\lim_{x\to+\infty}(f(x)-a_1\cdot x) = \lim_{x\to+\infty}\left(\sqrt{\frac{x^3}{x-1}}-x\right) = \begin{bmatrix}\text{Smena}\\ t=\frac{1}{x}\end{bmatrix} = \lim_{t\to0^+}\left(\sqrt{\frac{\frac{1}{t^3}}{(\frac{1}{t}-1)}}-\frac{1}{t}\right) = \lim_{t\to0^+}\left(\sqrt{\frac{\cancel{t}}{t^3(1-t)}}-\frac{1}{t}\right)$$

$$= \lim_{t\to0^+}\frac{1}{t}\left(\frac{1}{\sqrt{1-t}}-1\right) = \lim_{t\to0^+}\frac{1-\sqrt{1-t}}{t\sqrt{1-t}} = \lim_{t\to0^+}\underbrace{\frac{\sqrt{1-t}-1}{-t}}_{\to\frac{1}{2}}\cdot\underbrace{\frac{1}{\sqrt{1-t}}}_{\to1} = \boxed{\frac{1}{2} = b_1}$$

$y_1 = x + 1/2,\ x\to+\infty$

- $-\infty$: $\lim_{x\to-\infty}\frac{f(x)}{x} = \lim_{x\to-\infty}-\sqrt{\frac{x^3}{x^2(x-1)}} = \boxed{-1 = a_2}$

$$\lim_{x\to-\infty}(f(x)-a_2\cdot x) = \lim_{x\to-\infty}\left(-\sqrt{\frac{x^3}{(x-1)}}+x\right) = \lim_{x\to-\infty}x\left(-\sqrt{\frac{x}{x-1}}+1\right) = \lim_{x\to-\infty}\left(-\sqrt{1+\frac{1}{x-1}}+1\right) =$$

$$= \lim_{x\to-\infty}x\left(-1-\frac{1}{2}\cdot\frac{1}{x-1}+o\left(\frac{1}{x-1}\right)+1\right) = \lim_{x\to-\infty}\left(-\frac{x}{2(x-1)}+o\left(\frac{x}{x-1}\right)\right) = \lim_{x\to-\infty}\left(-1/2\cdot\underbrace{\frac{x}{x-1}}_{\to1}+\underbrace{o(1)}_{\to0}\right) = \boxed{-1/2 = b_2}$$

$y_2 = -x - 1/2,\ x\to-\infty$

↓Nastavak↓

$$9.\ f(x) = 2\ln|e^x-1| - \ln|e^x-2| = \underbrace{\ln|e^x-1| + \ln\left|\frac{e^x-1}{e^x-2}\right|}$$

$$\lim_{x\to+\infty}\left(\ln|e^x-1| + \ln\left|\frac{e^x-1}{e^x-2}\right|\right) = +\infty$$ Nema horizontalne

$$\lim_{x\to+\infty}\left(\frac{2\ln|e^x-1| - \ln|e^x-2|}{x}\right) = \lim_{x\to+\infty}\left(\frac{2\ln e^x\left|1-\frac{1}{e^x}\right| - \ln e^x\left|1-\frac{2}{e^x}\right|}{x}\right) = \lim_{x\to+\infty}\left(\frac{2\left(x+\overset{\to 0}{\ln\left|1-\overset{\to 0}{\frac{1}{e^x}}\right|}\right) - x - \overset{\to 0}{\ln\left|1-\overset{\to 0}{\frac{2}{e^x}}\right|}}{x}\right)$$

$$= \lim_{x\to+\infty}\frac{2x-x}{x} = \boxed{1 = a}$$

$$\lim_{x\to+\infty}(f(x)-a\cdot x) = \lim_{x\to+\infty}\left(2\cancel{x} + \overset{\to 0}{\ln\left|1-\underset{\to 0}{\frac{1}{e^x}}\right|} - \cancel{x} - \overset{\to 0}{\ln\left|1-\underset{\to 0}{\frac{2}{e^x}}\right|} - \cancel{x}\right) = \boxed{0 = b}$$ $y = x$, kada $x \to +\infty$

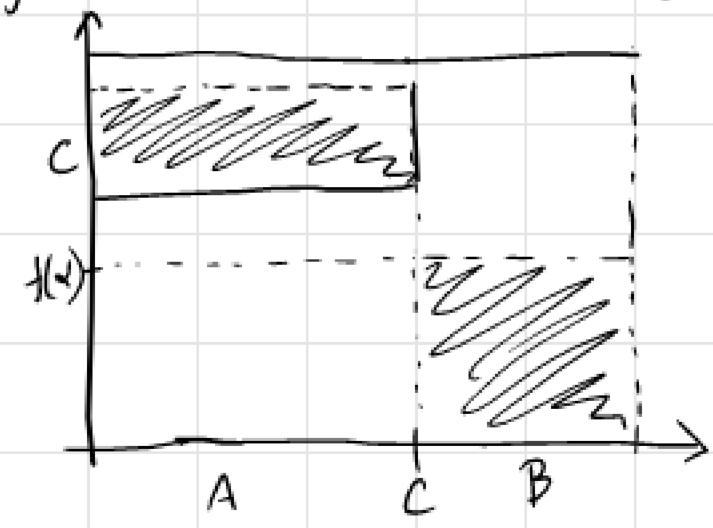
$$\lim_{x\to-\infty}\left(2\ln|\overset{\to 0}{e^x}-1| - \ln|\overset{\to 0}{e^x}-2|\right) = 2\ln 1 - \ln 2 = 0 - \ln 2 = -\ln 2$$ $y = -\ln 2$ Hor. asimptota

! $[0,1] = A \sqcup B$ A, B intervali B.G.O. $A = [0,c)$, $B = [c,1]$, $\underbrace{f(B) \subseteq A,\ f \text{ neprekidna } [c,1]^{\text{zatvoren interval}}}_{*}$

$\overset{\text{vajer}}{\overset{*}{\Rightarrow}}$ postiže max na $[c,1] \Rightarrow \exists \alpha \in [c,1]$ $f(\alpha) = \max f(x)$, $x \in [c,1]$

$f(B) \subseteq A \Rightarrow f(B) \subseteq [0,c) \Rightarrow f(\alpha) < c \Rightarrow \forall x \in [c,1]$ $f(x) \le f(\alpha) < c$

$f(A) \subseteq B \Rightarrow \forall x \in [0,c)$ $c \le f(x) \le 1$



Uzmemo $f(\alpha) < c \le f(a)$ $\forall a \in A$ $f(a) \ge c$

$$f(\alpha) < \frac{f(\alpha)+c}{2} < c \le f(a)$$

$$\left.\begin{array}{l} f(\alpha) < \frac{f(\alpha)+c}{2} < f(a) \\ f \text{ neprekidna na } [a,\alpha] \end{array}\right\} \xRightarrow{\text{T. o međuvrednosti}} \left.\begin{array}{l} \exists d \in (a,\alpha)\ f(d) = \frac{f(\alpha)+c}{2} \\ \text{ako je } d \in A \Rightarrow f(d) \ge c \end{array}\right\} \lightning$$

$\Rightarrow d \notin A$

$$\text{ako } d \in B \quad \left.\begin{array}{l} f(d) \le f(\alpha) \\ f(d) = \frac{f(\alpha)+c}{2} > f(\alpha) \end{array}\right\} \lightning \Rightarrow d \notin B$$

$\Rightarrow d \notin A,\ d \notin B,\ d \in [0,1]$ $\lightning$

7. čas

Ponavljanje $f: A \to \mathbb{R}, a \in A$

f je diferencijabilna ako postoji $\lim\limits_{x \to a} \frac{f(x)-f(a)}{x-a}$, $\lim \frac{f(a+h)-f(a)}{h}$

$f'(a) = \lim\limits_{h \to 0} \frac{f(a+h)-f(a)}{h}$

$f, g : A \to \mathbb{R}$

$(f(x)+g(x))' = f'(x) + g'(x)$

$\alpha \in \mathbb{R}$ $(\alpha f(x))' = \alpha f'(x)$

$(f(x) \cdot g(x))' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$

$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$

Neki tablični, $\alpha \in \mathbb{R}$

x^α	$\alpha x^{\alpha-1}$
e^x	e^x
$\ln x$	$\frac{1}{x}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\operatorname{arctg} x$	$1+x^2$

$a > 0$, $(a^x)' = (e^{x \ln a})' = e^{x \ln a} \cdot (x \ln a)' = e^{x \ln a} \cdot \ln a = a^x \ln a$

$(\log_a x)' = \left(\frac{\ln x}{\ln a}\right)' = \frac{1}{\ln a} \cdot \frac{1}{x}$

$(\arcsin x)' = ?$ $\quad \sin(\arcsin x) = x \;'$

$(\sin(\arcsin x))' = 1 \;\big|\; \sin'(\arcsin x) \cdot (\arcsin x)' = 1 \;\big|\; \cos(\arcsin x) \cdot (\arcsin x)' = 1$

$(\arcsin x)' = \frac{1}{\cos(\underbrace{\arcsin x}_{\alpha})} = \frac{1}{\sqrt{1-x^2}}$

$\sin\alpha = x$, $\quad \alpha \in [-\pi/2, \pi/2]$

$\cos\alpha = ?$ $\quad \cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-x^2}$

$(\operatorname{arctg} x)' = ?$ $\quad \operatorname{tg}(\operatorname{arctg} x) = x \,/\,'$ $\quad \operatorname{tg}'(\operatorname{arctg} x) \operatorname{arctg} x = 1$

$(\operatorname{tg}\varphi)' = \left(\frac{\sin\varphi}{\cos\varphi}\right)' = \frac{\cos\varphi \cdot \cos\varphi - \sin\varphi(-\sin\varphi)}{(\cos\varphi)^2} = \frac{\cos^2\varphi + \sin^2\varphi}{\cos^2\varphi} = \frac{1}{\cos^2\varphi}$

$\frac{1}{\cos^2(\operatorname{arctg} x)} \cdot (\operatorname{arctg} x)' = 1 \implies (\operatorname{arctg} x)' = \boxed{\overset{1}{\cos^2(\underbrace{\operatorname{arctg} x}_{\alpha})}} = \frac{1}{\sqrt{1+x^2}}$

$\operatorname{tg}\alpha = x$ $\quad \cos^2 x = ?$ preko $\operatorname{tg}\alpha$

$\cos^2\alpha = \frac{1}{1+x^2}$ $\quad \frac{\sin^2\alpha}{\cos^2\alpha} = x^2$ $\quad \frac{1-\cos^2\alpha}{\cos^2\alpha} = x^2$ $\quad \boxed{\frac{1}{\cos^2\alpha} - 1 = x^2}$

$(f \cdot g \cdot h)'(x) = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$

$(f \circ g \circ h)'(x) = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x)$

① $f(x) = 2x^3 e^x \ln x$

$f'(x) = 2(3x^2 e^x \ln x + x^3 e^x \ln x + x^3 e^x \frac{1}{x}) = 2e^x x^2(3\ln x + x\ln x + 1)$

② $f(x) = \sqrt{1 + \sqrt[3]{1+x^4}} = (1 + (1+x^4)^{1/3})^{1/2} \,/\,'$

$f'(x) = \frac{1}{2}(1 + (1+x^4)^{1/3})^{-1/2} = \frac{1}{3}(1+x^4)^{-2/3} \cdot 4x^3$ $\quad$ $g(x) = \sqrt{x}$, $h(x) = 1 + \sqrt[3]{x}$, $\mathcal{K}(x) = 1 + x^4$ $\quad f(x) = (g \circ h \circ \mathcal{K})(x)$

↓Nastavak↓

③ $f(x)=x^x=e^{x\ln x}$

$f'(x)=e^{x\ln x}\left(1\cdot\ln x+x\cdot\frac{1}{x}\right)=x^x(1+\ln x)$

$$\lim_{h\to 0}\frac{\arcsin(x+h)-\arcsin x}{h}=\left[\begin{matrix}\arcsin(x+h)=\beta\\ \arcsin x=\alpha\end{matrix}\right]=$$

$\arcsin x=\alpha$, $\sin\alpha=x\to\cos\alpha=\sqrt{1-x^2}$

$\arcsin(x+h)=\beta$, $\sin\beta=x+h\to\cos\beta=\sqrt{1-(x+h)^2}$

$\sin(\beta-\alpha)=\sin\beta\cos\alpha-\sin\alpha\cos\beta=(x+h)\sqrt{1-x^2}-x\sqrt{1-(x+h)^2}$

$$=\lim_{h\to 0}\frac{\sin(\beta-\alpha)}{h}\cdot\underbrace{\frac{\beta-\alpha}{\sin(\beta-\alpha)}}_{\to 1}=\lim_{h\to 0}\frac{\sin(\beta-\alpha)}{h}=\lim_{h\to 0}\frac{(x+h)\sqrt{1-x^2}-x\sqrt{1-(x+h)^2}}{h}=$$

$$=\lim_{h\to 0}\left(\frac{\frac{x(1-x^2-1+(x+h)^2)}{\sqrt{1-x^2}+\sqrt{1-(x+h)^2}}}{h}+\sqrt{1-x^2}\right)=\lim_{h\to 0}\left(\frac{x(2xh+h^2)}{h(\sqrt{\ }+\sqrt{\ })}+\sqrt{1-x^2}\right)=\lim_{h\to 0}\left(\frac{2x^2+xh}{\sqrt{1-x^2}+\sqrt{1-\underset{\to x^2}{(x+h)^2}}}+\sqrt{1-x^2}\right)$$

$$=\frac{\cancel{2}x^2}{\cancel{2}\sqrt{1-x^2}}+\sqrt{1-x^2}=\frac{1}{\sqrt{1-x^2}}$$

④ $f(x)=x^{x^x}$

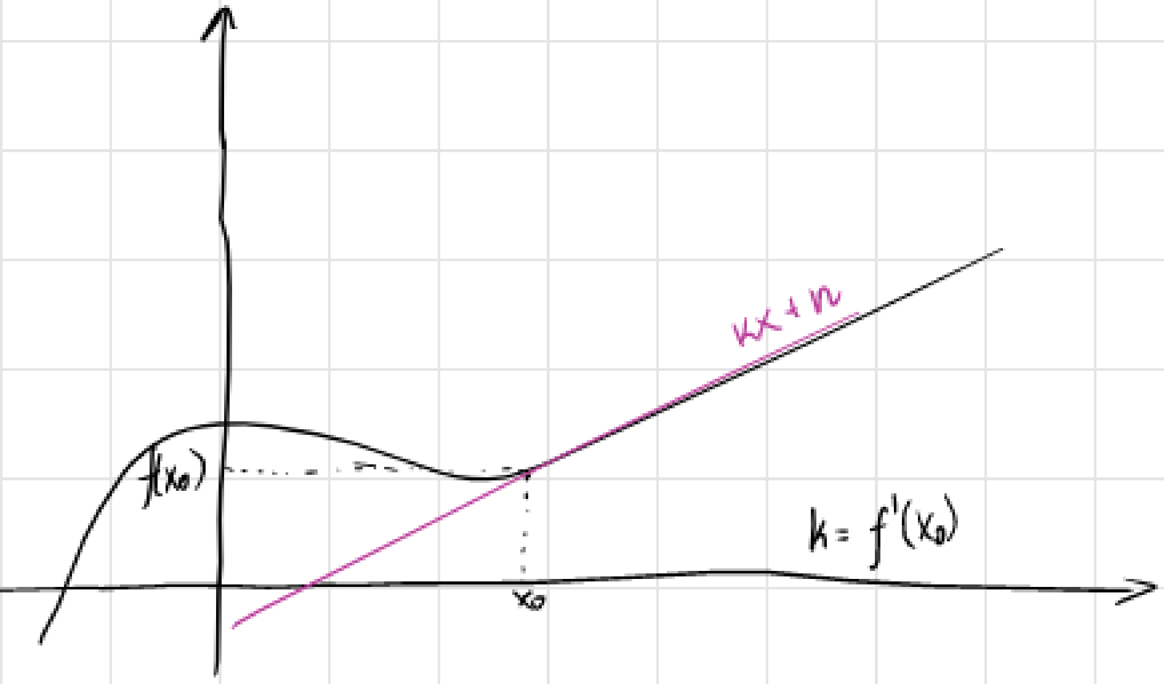
$f(x)=e^{x^x\cdot\ln x}\ /'$ $\quad f'(x)=e^{x^x\ln x}\left(x^x(1+\ln x)+x^x\cdot\frac{1}{x}\right)=x^{x^x}\left(x^x(1+\ln x)+x^{x-1}\right)$

$=x^{x^x}\cdot x^x\left(1+\ln x+\frac{1}{x}\right)$

Domaći: ⑤ $f(x)=\left(\frac{\arcsin(\sin^2 x)}{\arccos(\cos^2 x)}\right)^{\operatorname{arctg}x}$, $f'(x)=?$

⑥ $f(x)=(\operatorname{arctg}x)^{x^2}=e^{x^2\cdot\ln(\operatorname{arctg}x)}$

$f'(x)=e^{x^2\cdot\ln(\operatorname{arctg}x)}\cdot\left(2x\ln(\operatorname{arctg}x)+x^2\cdot\frac{1}{\operatorname{arctg}x}\cdot\frac{1}{1+x^2}\right)$



⑦ $f(x)=3x^2+e^x-x$ tangenta u $x_0=1$

$y=kx+n$? $\quad f(x_0)=k\cdot x_0+n$

$k=f'(x_0)$

$f'(x)=6x+e^x-1 \qquad f'(1)=6+e-1=5+e$

$y=(5+e)x+n \qquad f(1)=(5+e)\cdot 1+n$

$2+\cancel{e}=5+\cancel{e}+n\Rightarrow n=-3$ tangenta $y=(5+e)x-3$

⑧ $a, b \in \mathbb{R}$ tangenta f diferencijabilna na $\mathbb{R}$

$$f(x)=\begin{cases} x\cdot e^{-bx}+a, & x<0 \\ x^3+bx, & x\geq 0 \end{cases}$$

, f neprekidna

$$\left.\begin{aligned} \lim_{x\to 0+} f(x) &= \lim_{x\to 0+}(x^3+bx) = 0 = f(0) \\ \lim_{x\to 0-} f(x) &= \lim(x\cdot e^{-bx}+a) = a \end{aligned}\right\} \quad \begin{aligned} f(0)&=a \\ a&=0 \end{aligned}$$

dif na $\mathbb{R}\setminus\{0\}$ fja je svakako dif.

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x} \; ; \left.\begin{aligned} \lim_{x\to 0^+}\frac{x^3+bx-0}{x} &= \lim_{x\to 0^+}(x^2+b) = \underline{\underline{b}} \\ \lim_{x\to 0^-}\frac{x e^{-bx}}{x} &= \lim_{x\to 0^-}(e^{-bx}) = \underline{\underline{1}} \end{aligned}\right\}$$

Za $b=1$ postoji $\lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = f'(0) = 1$

$b=1$ i $a=0$ fja f je dif.

Dodatno objašnjenje:

$(x^3+bx)' = 3x^2+b = g'(x)$ $\quad b = g'(0)$ $\quad \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = 3x^2+b$

$(x\cdot e^{-bx})' = -bxe^{-bx}+e^{-bx} = h'(x)$ $\quad 1 = h'(0)$

⑨ $f(x)=\sqrt{1-e^{-x^2}}$ ispitati dif f-je f na njenom domenu

$1-e^{-x^2}\geq 0 \quad 1\geq e^{-x^2} \iff -x^2\leq 0 \;\forall x \quad \mathcal{D}_f = \mathbb{R}$ f je neprekidna

$f'(x) = \frac{1}{2}\cdot\frac{1}{\sqrt{1-e^{-x^2}}}\cdot(-e^{-x^2}\cdot(-2x)) = \frac{1}{2}\cdot\frac{2xe^{-x^2}}{\sqrt{1-e^{-x^2}}}$ nije definisana u 0! Tražimo ručno

$$f'(0) = \lim_{h\to 0}\frac{f(h)-f(0)}{h} = \lim_{h\to 0}\frac{\sqrt{1-e^{-h^2}}-0}{h} = \lim_{h\to 0}\operatorname{sgn}(h)\sqrt{\frac{1-e^{-h^2}}{h^2}} = \lim_{h\to 0}\operatorname{sgn}(h)\sqrt{\frac{1-1+h^2+o(h^2)}{h^2}} =$$

$$= \lim_{h\to 0}\operatorname{sgn}(h)\sqrt{1+o(1)} = \lim_{h\to 0^+}\frac{f(h)-f(0)}{h} = 1 \neq \lim_{h\to 0^-}\frac{f(h)-f(0)}{h} = -1$$

$\Rightarrow \lim_{h\to 0}\frac{f(h)-f(0)}{h}$ ne postoji $\Rightarrow f$ nije dif

⑩ $f(x) = x\cdot|x|$ $\quad f$ dif. na $\mathbb{R}\setminus\{0\}$

$\lim_{h\to 0+}\frac{f(h)-f(0)}{h} = \lim_{h\to 0^+}\frac{h\cdot h}{h} = 0^{*}$ $\quad \lim_{h\to 0^-}\frac{f(h)-f(0)}{h} = \lim_{h\to 0^-}\frac{-h^2}{h} = 0^{*}$ $\Rightarrow f$ dif na $\mathbb{R}$

$\lim_{h\to 0}\frac{f(h)-f(0)}{h} = \lim_{h\to 0}\frac{\cancel{h}\,|h|^{\to 0}}{\cancel{h}} = 0$ $\quad \lim_{h\to 0}\frac{f(h)-f(0)}{h} = 0 \Rightarrow f$ dif u 0

⑪ $f(x) = |x^2-x-12|\cos\frac{x\pi}{2} = |(x-4)(x+3)|\cos\frac{x\pi}{2}$

$(x+3)(x-4)\cos\frac{x\pi}{2} \quad -(x+3)(x-4)\cos\frac{x\pi}{2}$

$-3 \quad 4$

$x=-3$: $\lim_{x\to -3}\frac{f(x)-f(-3)}{x+3} = ?$

$$\left.\begin{aligned} \lim_{x\to -3^+}\frac{f(x)-f(-3)}{x+3} &= \lim_{x\to -3^+}\frac{-\cancel{(x+3)}(x-4)\cos\frac{x\pi}{2}}{\cancel{x+3}} = -(-7)\cdot 0 = 0 \\ \lim_{x\to -3^-}\frac{f(x)-f(-3)}{x+3} &= \lim_{x\to -3^-}\frac{\cancel{(x+3)}(x-4)\cos\frac{x\pi}{2}}{\cancel{x+3}} = -7\cdot 0 = 0 \end{aligned}\right\} \begin{aligned} &\lim_{x\to -3}\frac{f(x)-f(-3)}{x+3} = 0 \\ &\Rightarrow f \text{ dif u } x=-3 \end{aligned}$$

$$\lim_{x\to 4^+}\frac{f(x)-f(4)}{x-4} = \lim_{x\to 4^+}\frac{(x+3)\cancel{(x-4)}\cos\frac{x\pi}{2}^{\nearrow 1}}{\cancel{(x-4)}} = 7 \quad \lim_{x\to 4^+}\frac{f(x)-f(4)}{x-4} = \lim_{x\to 4^+}\frac{-(x+3)\cancel{(x-4)}\cos\frac{x\pi}{2}^{\nearrow 1}}{\cancel{(x-4)}} = -7$$

$\Longrightarrow$ Ne postoji $\lim_{x\to 4}\frac{f(x)-f(4)}{x-4} \Rightarrow f$ nije dif u $x=4$

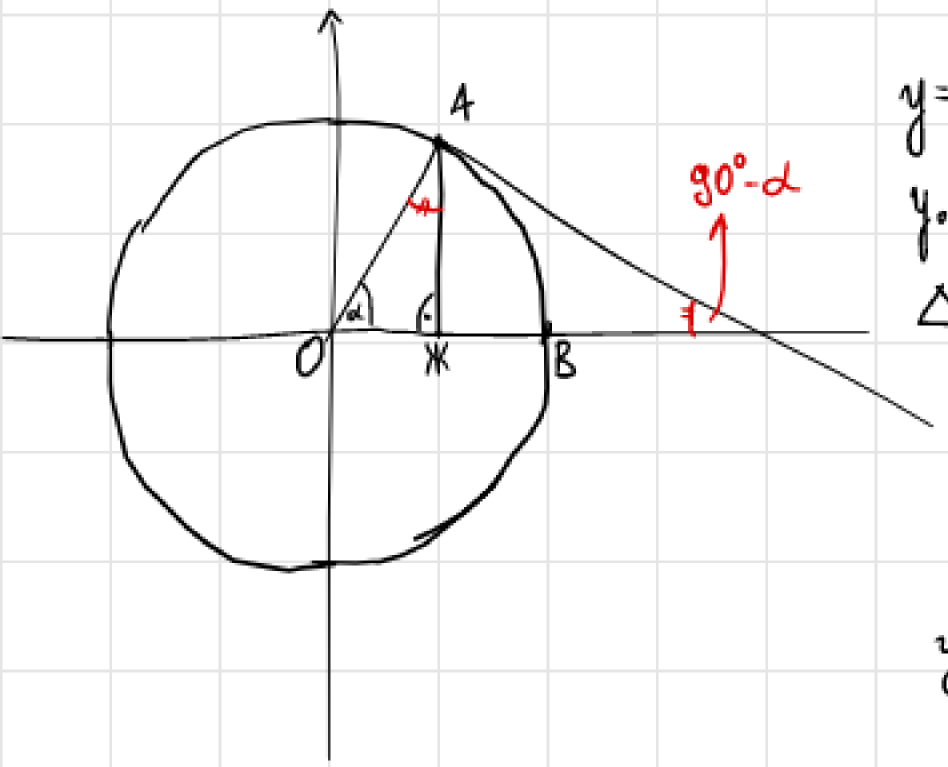
⑫ $f(x) = [x]\sin \pi x \quad , x \in \mathbb{R} \setminus \mathbb{Z}$

$$f'(x) = \underset{\nwarrow \text{const}}{[x]} \cdot \pi \cos \pi x$$

$k \in \mathbb{Z}$

$$\lim_{h \to 0^+} \frac{f(k+h) - f(k)}{h} = \lim_{h \to 0^+} \frac{k\sin(\pi(k+h)) - k\sin \pi k}{h} = \lim_{h \to 0^+} \frac{k\sin(k\pi + h\pi)}{h} = \lim_{h \to 0^+} \frac{k(-1)^k \cdot \overset{\nearrow 1}{\boxed{\frac{\sin(\pi h)}{\pi h}}} \cdot \pi}{} = \pi \cdot k \cdot (-1)^k$$

$$\lim_{h \to 0^-} \frac{f(k+h) - f(k)}{h} = \lim_{h \to 0^-} \frac{k\sin(\pi(k+h)) - k\sin \pi k}{h} = \lim_{h \to 0^-} \frac{(k-1) \cdot \sin(k\pi + h\pi)}{h} = \pi(k-1)(-1)^k$$



$y = kx + n$

$y_0 = kx_0 + n$

$\triangle BЖA \sim \triangle OAЖ \quad k = \frac{AЖ}{BЖ} = \frac{OЖ}{AЖ} = \frac{x_0}{y_0}$

$y_0 = kx_0 + n$

$y = \frac{x_0}{y_0} + x + \frac{1}{y_0} \qquad y_0 = \frac{x_0^2}{y_0} + h$

$y = \frac{-x_0}{\sqrt{1 - x_0^2}} \cdot x + \frac{1}{\sqrt{1 - x_0^2}} \qquad n = \frac{y_0^2 + x_0^2}{y_0}$

$f(x) = \sqrt{1 - x^2} \qquad f'(x) = \frac{1}{2} \frac{-2x}{\sqrt{1 - x^2}} = \frac{-x}{\sqrt{1 - x^2}} \qquad f'(x_0) = -\frac{x_0}{\sqrt{1 - x_0^2}}$

$k = \frac{-x_0}{\sqrt{1 - x_0}}$

8. čas

Rolova teorema

$f:[a,b]\to\mathbb{R}$, f je neprekidna na $[a,b]$, diferencijabilna na (a,b)

$f(a)=f(b) \Rightarrow \exists\ c\in(a,b)\ f'(c)=0$

f dostiže lokalni ekstremum u x_0, f diferencijabilna u $x_0 \Rightarrow f'(x_0)=0$

Bez umanjenja opstosti u x_0 min.

$(\exists\varepsilon>0)\ |x-x_0|<\varepsilon \Rightarrow f(x)\geq f(x_0)$

$$\lim_{x\to x_0^+}\frac{f(x)-f(x_0)}{x-x_0}\geq 0 \qquad \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}\leq 0$$

Lagranžova teorema

$f:[a,b]\to\mathbb{R}$ f neprekidna na $[a,b]$, diferencijabilna na (a,b)

$\Rightarrow(\exists c\in(a,b))\ f'(c)=\frac{f(b)-f(a)}{b-a}$

$\underbrace{f'(c)-\frac{f(b)-f(a)}{b-a}}=0 \longrightarrow g'(c)$

$g'(x)=f'(x)-\frac{f(b)-f(a)}{b-a}$

$g(x)=f(x)-\frac{f(b)-f(a)}{b-a}\cdot x$

$g(a)=f(a)-\frac{f(b)-f(a)}{b-a}\cdot a=f(a)\left(1+\frac{a}{b-a}\right)-f(b)\frac{a}{b-a}=f(a)\cdot\frac{b}{b-a}-f(b)\frac{a}{b-a}$

$g(b)=f(b)-\frac{f(b)-f(a)}{b-a}\cdot b=f(b)\left(1-\frac{b}{b-a}\right)+f(a)\cdot\frac{b}{b-a}=f(a)\cdot\frac{b}{b-a}-f(b)\frac{a}{b-a}$

$\Rightarrow g(a)=g(b)$

g neprekidna na $[a,b]$, diferencijabilna na (a,b)

$\xRightarrow{Rol}(\exists c\in(a,b))\ g'(c)=0 \quad f'(c)-\frac{f(b)-f(a)}{b-a}=0 \quad f'(c)=\frac{f(b)-f(a)}{b-a}$ $\blacksquare$

1. $f:\mathbb{R}\to\mathbb{R}$, $(\exists a\in\mathbb{R})\ f(a)=1$, $a\neq0$

Dokazati da $\exists c\in\mathbb{R}$ t.d. $c\cdot f'(c)+f(c)=1$

$g'(x)=x\cdot f'(x)+f(x)\cdot 1-1 \qquad g(x)=x\cdot f(x)-x$

$g(x)=x(f(x)-1) \quad g(0)=0 \quad g(a)=(1-1)\cdot a=0,\ a\neq0 \quad g(0)=g(a)$

g neprekidna i dif

$\xRightarrow{Rol}$ ($\exists c$ između $0,a$) $g'(c)=0$ $(g(x)=xf(x)-x)$

$c\cdot f'(c)+f(c)-1=0$ tj. $c\cdot f'(c)+f(c)=1$

2. $(\forall x, y > 0)\ \left|\frac{1}{1+x} - \frac{1}{1+y}\right| \leq |x-y|$

$f(x) \leftarrow \left|\frac{\boxed{\frac{1}{1+x}} - \frac{1}{1+y}}{x-y}\right| \leq 1 \qquad \left|\frac{f(x)-f(y)}{x-y}\right| \leq 1$ za $f(x) = \frac{1}{1+x}$

f dif, nepr $\quad (\exists c$ između x i $y)\ f'(c) = \frac{f(x)-f(y)}{x-y}, \ |f'(c)| < 1 \Rightarrow \left|\frac{f(x)-f(y)}{x-y}\right| < 0$

$(\forall t > 0)\ f'(t) = \frac{-1}{(1+t)^2} = -\frac{1}{(1+t)^2} > -1 \Rightarrow -1 < f'(c) < 0 \Rightarrow |f'(0)| < 1$

3. $(\forall x, y \geq 1)\ \left|\frac{x}{1+e^{\frac{1}{x}}} - \frac{y}{1+e^{\frac{1}{y}}}\right| \geq \frac{|x-y|}{4}$

$x \neq y \ \left|\frac{f(x)-f(y)}{x-y}\right| \geq \frac{1}{4}$ za $f(t) = \frac{t}{1+e^{\frac{1}{t}}}$

f nepr, dif

$f'(t) = \frac{1+e^{\frac{1}{t}} - t \cdot e^{\frac{1}{t}}\left(-\frac{1}{t^2}\right)}{(1+e^{\frac{1}{t}})^2} = \frac{1+e^{1/t}+e^{1/t}\cdot\frac{1}{t}}{(1+e^{\frac{1}{t}})^2}$ Hoćemo da je $f'(t) \geq \frac{1}{4}$ za $t \geq 1$

$\frac{1+e^{\frac{1}{t}}(1+\frac{1}{t})}{(1+e^{1/t})^2} \geq \frac{1}{4} \qquad \frac{1}{1+\underbrace{\boxed{e^{1/t}}}_{\leq e}} + \frac{1}{t} \cdot \frac{e^{1/t}}{(1+e^{1/t})^2} \geq \frac{1}{1+e} + \overbrace{\boxed{\frac{1}{t} \cdot \frac{e^{1/4}}{(1+e^{1/t})^2}}}^{>0} > \frac{1}{1+e} > \underbrace{\frac{1}{1+3}}_{e<3} = \frac{1}{4}$

f neprekidna, diferencijabilna

$x, y \geq 1$, $x \neq y$

$\left.\begin{array}{l} \xrightarrow{\text{Lagranžova}} (\exists c \text{ između } x \text{ i } y)\ f'(c) = \frac{f(x)-f(y)}{x-y} \\ \underline{f'(t) > \frac{1}{4}} \text{ za } \forall t \geq 1 \Rightarrow f'(c) > \frac{1}{4} \end{array}\right\} \frac{f(x)-f(y)}{x-y} > 1/4$

Za $x \neq y$ i $x, y \geq 1 \quad (\forall x, y \geq 1)$

$|f(x) - f(y)| \geq 1/4\, |x-y| \qquad \left|\frac{x}{1+e^{1/x}} - \frac{y}{1+e^{1/y}}\right| \geq \frac{|x-y|}{4}$

☆ ⇒ $|\operatorname{arctg} x - \operatorname{arctg} y| \leq |x-y|,\ x, y \in \mathbb{R}$

④ $f : [0,1] \to \mathbb{R}$ nepr na $[0,1]$ dif na $(0,1)$

Dokazati da $(\exists c \in (0,1))\ f(1) - f(0) = \frac{1}{2}\overbrace{(f'(c) + f'(1-c))}^{g'(c)}$

$g'(x) = f'(x) + f'(1-x) \qquad g(x) = f(x) - f(1-x)$

$g(1) = f(1) - f(0) \qquad g(0) = f(0) - f(1)$

g nepr. dif. $\xrightarrow{\text{Lagranž}} (\exists c \in (0,1))\ g'(c) = \frac{g(1)-g(0)}{1-0}$

$g'(c) = \frac{f(1) - f(0) - f(0) + f(1)}{1} = 2(f(1) - f(0))$

$f'(c) + f'(1-c) = 2(f(1) - f(0)) \Longrightarrow \frac{1}{2}(f'(0) + f'(1-c)) = f(1) - f(0)$

5. $f, g : [a,b] \to \mathbb{R}$ $\quad f, g$ nepr na $[a,b]$, dif na (a,b)

$f(a) = f(b) = 0 \quad g(a)\, g(b) \neq 0$

$(\forall x \in (a,b))\ f'(x) g(x) \neq f(x) g'(x)$

Dokazati da $(\exists c \in (a,b))\ g(c) = 0$

$f'(x) g(x) - f(x) g'(x) \neq 0 \quad (*)$

PPS $g(x) \neq 0$ ni za jedno $x \in (0,1)$

$h(x) = \frac{f(x)}{g(x)} \leftarrow$ definisana, nepr, dif

$h(a) = \frac{f(a)}{g(a)} \quad h(b) = 0$

$\xRightarrow{Rol}$ $(\exists c \in (a,b))\ h'(c) = 0$

$\frac{f'(x) g(c) - g'(c) f(c)}{g^2(c)} = 0 \Rightarrow f'(x) g(x) - f(x) g'(x) = 0$

↯ sa $*$

6. f nepr na $[a,b]$, dif na (a,b), $f(x) \neq 0$ za $\forall x \in (a,b)$

Dokazati da $\exists c \in (a,b)$

$\frac{f'(c)}{f(c)} = \frac{1}{a-c} + \frac{1}{b-c} \qquad f'(c) = \frac{f(c)}{a-c} + \frac{f(c)}{b-c} \qquad f'(c) - f(c)\left(\frac{1}{a-c} + \frac{1}{b-c}\right) = 0$

$(a-c)(b-c)\, f'(c) = f(c)\,(b-c+a-c)$

$(ab - c(a+b) + c^2) f'(c) = f(c)(a+b) - 2c f(c)$

$\underline{ab f'(x)} - \underline{(a+b) x f'(x)} + \underline{x^2 f'(x)} - \underline{f(x)(a+b)} + \underline{2x f(x)} = g'(x)$

$g(x) = ab f(x) - (a+b) x f(x) + x^2 f(x)$

$g(a) = ab f(a) - (a+b) a f(a) + a^2 f(a) = (ab - a^2 - ab + a^2)\, f(a) = 0$

$g(b) = ab f(b) - (a+b) b f(b) + b^2 f(b) = 0$

$g \to$ nepr, dif $\quad g(a) = g(b) \xRightarrow{Rol} (\exists c \in (a,b)) \quad g'(c) = 0$

$g'(c) = ab f'(c) - (a+b) c f'(c) - (a+b) f(c) + c^2 f'(c) + 2c f(c) = 0$

$\Rightarrow f'(c)(ab - (a+b)c + c^2) = (a+b-2c) f(c)$

$\xRightarrow{f(c) \neq 0} \frac{f'(c)}{f(c)} = \frac{a+b-2c}{ab - ac - ac + c^2} = \frac{1}{a-c} + \frac{1}{b-c}$

Lopitalova pravila

$f, g : (a,b) \to \mathbb{R} \quad -\infty < a < b \leq +\infty$

f, g dif. $\frac{0}{0}, \frac{\infty}{\infty}$

$\lim_{x \to a^+} f(x) = \lim_{x \to a^+} g(x) = 0$

ili $\lim_{x \to a^+} g(x) = \infty$

Tada važi ako postoji $\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L \in \mathbb{R} \Rightarrow \lim_{x \to a^+} \frac{f(x)}{g(x)} = L$

* ~~$\lim_{x \to a^+} \frac{f(x)}{g(x)} \overset{LOP}{=} \lim_{x \to a^+} \frac{f'(x)}{g'(x)} = \ldots\ldots$~~ $=$ ne postoji Onda biramo drugi metod

① $\lim_{x \to +\infty} \left(\frac{2}{\pi} \operatorname{arctg} x\right)^x = \lim_{x \to +\infty} e^{x \ln\left(\frac{2}{\pi} \operatorname{arctg} x\right)} = \lim_{x \to +\infty} \left(x \ln\left(\frac{2}{\pi} \operatorname{arctg} x\right)\right)$

$= \lim_{x \to +\infty} \frac{\ln\left(\frac{2}{\pi} \operatorname{arctg} x\right)}{\frac{1}{x}} \overset{LOP}{=} \lim_{x \to +\infty} \frac{\frac{1}{\frac{2}{\pi} \operatorname{arctg} x} \cdot \frac{1}{1+x}}{\frac{-1}{x^2}}$

$= \lim_{x \to +\infty} \frac{\frac{\pi}{2}}{\operatorname{arctg} x} \cdot \frac{-x^2}{1+x^2} = - \frac{\frac{\pi}{2}}{\frac{\pi}{2}} \cdot 1 = -1 \quad (*)$

2\. $\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x}} \overset{LOP}{=} \lim_{x \to 0^+} \frac{\frac{1}{x}}{-1/x^2} = \lim_{x \to 0^+} (-x) = 0$

3\. $\lim_{x \to 0} \frac{(a+x)^x - a^x}{x^2}, \; a > 0 \qquad ((a+x)^x)' = \left(e^{x \ln(a+x)}\right) = e^{x \ln(a+x)}\left(x \frac{1}{a+x} + \ln(a+x)\right)$

$\overset{LOP}{=} \lim_{x \to 0} \frac{(a+x)^x \left(\frac{x}{a+x} + \ln(a+x)\right) - a^x \ln a}{2x} \overset{LOP}{=} \frac{(a+x)^x \left(\frac{x}{a+x} + \ln(a+x)\right)^2 + (a+x)^x \left(-\frac{-a}{(a+x)^2} + \frac{1}{a+x}\right) - a^x \ln^2 a}{2}$

$= \lim_{x \to 0} \frac{1}{2}\left((a+x)^x \left(\left(\underset{\to 0}{\boxed{\frac{x}{a+x}}} + \underset{\to \ln a}{\boxed{\ln(a+x)}}\right)^2 + \underset{\to \frac{1}{a}}{\boxed{\frac{a}{(a+x)^2}}} + \underset{\to 1/a}{\boxed{\frac{1}{a+x}}}\right) - \underset{\to 1}{\boxed{a^x}} \ln^2 a\right) =$

$= 1/2 \left(1 \cdot \left(\ln^2 a + \frac{2}{a}\right) - 1 \cdot \ln^2 a = \frac{1}{a}\right.$

Poredbena skala

$\lim_{x \to +\infty} \frac{\ln x}{x^\alpha} = 0$ za bilo koje $\alpha > 0$

$\lim_{x \to +\infty} \frac{x^\alpha}{a^x} = 0$ za bilo koje $\alpha > 0, a > 0$ $\qquad \lim_{x \to +\infty} a^x \cdot x^\alpha = 0, \; a < 0$

$\lim_{x \to +\infty} \frac{x^{500} + e^{\frac{x}{20}} + \log^{1000} x}{2x^{1000} + e^{x^2}} = \lim_{x \to \infty} \frac{e^{\frac{x}{20}}}{e^{x^2}} = 0$

$$\lim_{x\to+\infty} \frac{\log_2 x - x^{700} + (\sqrt{3})^x + \sin(x^4-x)}{x^{2020} - 2(\sqrt{3})^x + \log_5 x} = \lim_{x\to+\infty} \frac{\underbrace{\frac{\log_2 x}{\sqrt{3}^x}}_{\to 0} - \underbrace{\frac{x^{700}}{\sqrt{3}^x}}_{\to 0} + 1 + \underbrace{\frac{\sin(x^4-x)}{(\sqrt{3})^x}}_{\to 0}}{\underbrace{\frac{x^{2020}}{\sqrt{3}^x}}_{\to 0} - 2 + \underbrace{\frac{\log_5 x}{\sqrt{3}^x}}_{\to 0}} = -\frac{1}{2}$$

Čas 9

f tačka u $a \in \mathbb{R}$

$$T'_{f,a,n}(x) = \sum_{k=0}^{n} \frac{f^{(n)}(a)}{k!}(x-a)^k$$

$$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)\cdot(x-a)^2}{2!} \ldots\ldots$$

$$f(x) = \sum_{n=0}^{n} \frac{f^{(n)}(a)\cdot(x-a)^k}{k!} + o((x-a)^n), x\to a$$

$$M_{f,n}(x) = \sum_{k=0}^{n} \frac{f^{(n)}(0)}{k!} x^k$$

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k + o(x^n), x\to 0$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

$$\sin x = \underbrace{\sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!}}_{} + o(x^{2n+2}), x\to 0$$

polinom $2n+1$
polinom $2n+2$ stepena

$$\cos x = 1 - \frac{x^2}{2} + o(x^3), x\to 0$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \ldots$$

$$\cos x = \sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!} + o(x^{2n+1}), x\to 0$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$e^x = \sum_{k=0}^{n} \frac{x^k}{k!} + o(x^n), x\to 0$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

$$\ln(1+x) = \sum_{k=1}^{n} (-1)^{k+1} \frac{x^k}{k} + o(x^n), x\to 0$$

$$(1+x)^{\alpha} = 1+\alpha x+o(x)$$

$$(1+x)^{\alpha} = \sum_{k=0}^{n} \binom{\alpha}{k}\cdot x^{k} + o(x^{n}),\ x\to 0$$

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)(\alpha-2)\cdot\ldots\cdot(\alpha-k+1)}{k!}$$

$$\sqrt{1+x} = (1+x)^{1/2} = 1+\frac{1}{2}x-\frac{1}{8}x^{2}+o(x^{4}),\ x\to 0$$

$$\binom{1/2}{2} = \frac{1/2\,(-1/2)}{2} = -1/8$$

$$(1+x)^{-1} = \sum_{k=0}^{n} (-1)^{k}x^{k}+o(x^{n}),\ x\to 0 \qquad \binom{-1}{k} = \frac{(-1)(-2)\cdots(-1-k+1)}{k!}$$

$$\frac{1}{1+x} = 1-x+x^{2}-x^{3}+x^{4}-x^{5}+\ldots \qquad = (-1)^{k}$$

$$f(x) = \cos x\cdot e^{2x} \overset{\text{Makloren 4. stepena}}{=\!=} \left(1-\frac{x^{2}}{2!}+\frac{x^{4}}{4!}+o(x^{4})\right)\left(1+2x+\frac{4x^{2}}{2!}+\frac{8x^{3}}{3!}+\frac{16x^{4}}{4!}+o(x^{4})\right)$$

$$= 1+2x+2x^{2}+4/3\,x^{3}+2/3\,x^{4}-\frac{x^{2}}{2}-x^{3}-x^{4}+\frac{x^{4}}{24}+o(x^{4})$$

$$= 1+2x+3/2\,x^{2}+1/3\,x^{3}-\frac{7}{24}x^{4}+o(x^{4}),\ x\to 0$$

$$M_{f,4}(x) = 1+2x+3/2\,x^{2}+1/3\,x^{3}-\frac{7}{24}x^{4}$$

$$\cos x\cdot e^{2x} = \left(1-\frac{x^{2}}{2}+o(x^{2})\right)\left(1+2x+2x^{2}+o(x^{2})\right) = 1+2x+\frac{3}{2}x^{2}+o(x^{2})$$

$$e^{\sin x} \overset{\text{Makloren 4. stepena}}{=\!=} e^{\overbrace{x-\frac{x^{3}}{3!}+o(x^{4})}^{t}} \qquad \cdot\, o\left(\left(x-\frac{x^{3}}{3!}+o(x^{4})\right)^{4}\right) = o(x^{4})$$

$$= 1+t+\frac{t^{2}}{2}+\frac{t^{3}}{3!}+\frac{t^{4}}{4!}+o(t^{4})$$

$$= 1+x-\frac{x^{3}}{6}+\frac{x^{2}}{2}-\frac{x^{4}}{6}+\frac{x^{3}}{6}+\frac{x^{4}}{24}+o(x^{4})$$

$$= 1+x+\frac{x^{2}}{2}-\frac{x^{4}}{8}+o(x^{4}),\ x\to 0$$

$$\sin x\cdot \ln(1+x) = \left(x-\frac{x^{3}}{6}+o(x^{3})\right)\left(x-\frac{x^{2}}{2}+\frac{x^{3}}{3}+o(x^{3})\right)$$

$$= x^{2}\underline{\left(1-\frac{x^{2}}{6}+o(x^{2})\right)\left(1-\frac{x}{2}+\frac{x^{2}}{3}+o(x^{2})\right)} \leftarrow \text{1 stepen}$$

$$= x^{2}\left(1+o(x)\right)\left(1-\frac{x}{2}+o(x)\right)$$

$$= x^{2}-\frac{x^{3}}{2}+o(x^{3})$$

* $f(x) = \operatorname{arctg} x$

$T_{f,\sqrt{3},2}(x) = ?$

$f(x) = \operatorname{arctg} x$ $\qquad f(\sqrt{3}) = \pi/3$

$f'(x) = \frac{1}{1+x^2}$ $\qquad f'(\sqrt{3}) = 1/4$

$f''(x) = (-1)(1+x^2)^{-2} \cdot 2x$ $\qquad f''(\sqrt{3}) = -\frac{2\sqrt{3}}{16} = -\frac{\sqrt{3}}{8}$

$$T_{f,\sqrt{3},2}(x) = \pi/3 + 1/4(x-\sqrt{3}) - \frac{\frac{\sqrt{3}}{8}}{\frac{2!}{1}} \cdot (x-\sqrt{3})^2$$

* $\operatorname{tg} x \to$ Maklorena 5. stepena

$$\operatorname{tg} x = \frac{\sin x}{\cos x} = \sin x \cdot \cos x^{-1} = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + o(x^5)\right)\left(1 - \overset{\nearrow t}{\boxed{\frac{x^2}{2!} + \frac{x^4}{4!} + o(x^5)}}\right)^{-1}$$

$$\left[(1+t)^{-1} = 1 - t + t^2 - t^3 + t^4 \qquad t = -\frac{x^2}{2} + \frac{x^4}{24} + o(x^4)\right]$$

$$= x\left(1 - \frac{x^2}{6} + \frac{x^4}{120} + o(x^4)\right)\left(1 + \frac{x^2}{2} - \frac{x^4}{24} + \frac{x^4}{4} + o(x^4)\right)$$

$$= x\left(1 + \frac{x^2}{2} + \frac{5}{24}x^4 - \frac{x^2}{6} - \frac{x^4}{120} + o(x^4)\right)$$

$$= x\left(1 + \frac{x^2}{3} + \frac{2}{15}x^4 + o(x^4)\right)$$

$$= x + \frac{x^3}{3} + \frac{2}{15}x^5 + o(x^5), x \to 0$$

1. $$\lim_{x\to 0} \frac{e^{\sin x} \cdot \cos(\sin x) - 1 - x}{x^3}$$

$$e^{\sin x} = 1 + x + \frac{x^2}{2} + o(x^3)$$

$$\cos(\sin x) = \cos x\left(x - \frac{x^3}{6} + o(x^3)\right) = 1 - \frac{\left(x - \frac{x^3}{6} + o(x^3)\right)^2}{2} + o\left(\left(x - \frac{x^3}{6} + o(x^3)\right)^3\right)$$

$$= 1 - \frac{x^2}{2} + o(x^3)$$

$$e^{\sin x}\cos(\sin x) = \left(1 + x + \frac{x^2}{2} + o(x^3)\right)\left(1 - \frac{x^2}{2} + o(x^3)\right)$$

$$= 1 - \frac{x^2}{2} + x - \frac{x^3}{2} + \frac{x^2}{2} + o(x^3) = 1 + x - \frac{x^3}{2} + o(x^3), x \to 0$$

$$\Rightarrow \lim_{x\to 0} \frac{e^{\sin x}\cos(\sin x)}{x^3} = \frac{1 + x - \frac{x^3}{2} + o(x^3) - 1 - x}{x^3} = -1/2$$

② $\lim\limits_{x\to 0} \dfrac{\sqrt{\cos x} - \sqrt[3]{\cos x}}{\sin^2 x}$ $\rightarrow (x+o(x))^2 = x^2 + o(x^2)$

Razvijamo do drugog stepena

Domaći!

③ $\lim\limits_{x\to 0} \dfrac{\ln(1+\sin x) + e^{-x} - 1}{\ln(1+x^2) - x\sin x}$

$$\ln(1+x^2) = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + o(x^8)$$

$$x\sin x = x\left(x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5)\right) = x^2 - \frac{x^4}{6} + \frac{x^6}{120} + o(x^6)$$

* Skratiće se

$$\ln(1+x^2) - x\sin x = \boxed{-\frac{x^4}{3} + o(x^4)}$$

Da smo ga razvili samo do drugog: $\left.\begin{array}{l} x\sin x = x^2 + o(x^2) \\ \ln(1+x^2) = x^2 + o(x^2) \end{array}\right\} o(x^2)$

$$\ln(1+\sin x) = \ln\left(1 + x - \frac{x^3}{6} + o(x^5)\right) = x - \frac{x^3}{6} + o(x^5) - \frac{\left(x - \frac{x^3}{6} + o(x^4)\right)^2}{2} + \frac{\left(x - \frac{x^3}{6} + o(x^4)\right)^3}{3}$$

$$- \frac{\left(x - \frac{x^3}{6} + o(x^4)\right)^4}{4} + o\left(\left(x - \frac{x^3}{6} + o(x^5)\right)^4\right)$$

$$= x - \frac{x^3}{6} - \frac{x^2}{2} + \frac{x^4}{6} + \frac{x^3}{3} - \frac{x^4}{4} + o(x^4)$$

$$= x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + o(x^4)$$

$$e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + o(x^4)$$

$$\lim_{x\to 0} \frac{\ln(1+\sin x) + e^{-x} - 1}{\ln(1+x^2) - x\sin x} = \lim_{x\to 0} \frac{\cancel{x} - \cancel{\frac{x^2}{2}} + \cancel{\frac{x^3}{6}} - \frac{x^4}{12} + \cancel{1} \cancel{- x} + \cancel{\frac{x^2}{2}} - \cancel{\frac{x^3}{6}} + \frac{x^4}{24} - \cancel{1} + o(x^4)}{-\frac{x^4}{3} + o(x^4)}$$

$$= \frac{-\frac{x^4}{24} + o(x^4)}{-\frac{x^4}{3} + o(x^4)} = \frac{-1/24}{-1/3} = 1/8$$

f-ja, tejlor + triv. limes, + Rol, lagranž teorijsui, niz ili supremum
$$\underset{\text{iskoristi}}{\downarrow}$$

④
$$\lim_{x\to+\infty}\left(\sqrt[5]{x^5+5x^4}-\sqrt{x^2-2x}\right)=\lim_{x\to+\infty} x\left(\sqrt[5]{1+\frac{5}{x}}-\sqrt{1-\frac{2}{x}}\right)=\ulcorner t=1/x,\ t\to 0 \lrcorner =$$
$$=\lim_{t\to0^+}\frac{\left(\sqrt[5]{1+5t}-\sqrt{1-2t}\right)}{t}=\lim_{t\to0^+}\frac{1+1/5\cdot5t+o(t)-1+\frac{1}{2}\cdot2t+o(t)}{t}$$
$$=\lim_{t\to0^+}\frac{2t+o(t)}{t}=2$$

b)
$$\lim_{x\to+\infty}\left(\sqrt[5]{x^{10}+5x^9}-\sqrt{x^4+2x^3}\right)=\lim_{x\to+\infty} x^2\left(\sqrt[5]{1+\frac{5}{x}}-\sqrt{1+\frac{2}{x}}\right)=\ulcorner t=1/x,\ t\to 0 \lrcorner =$$
$$\binom{1/2}{2}=-1/8$$
$$=\lim_{t\to0^+}\frac{\left(\sqrt[5]{1+5t}-\sqrt{1+2t}\right)}{t^2}=\lim_{t\to0^+}\frac{(1+5t)^{1/5}-(1+2t)^{1/2}}{t^2}$$
$$=\lim_{t\to0^+}\frac{1+1/3\cdot5t-\frac{2}{25}(5t)^2-1-1/2\cdot2t+1/8(2t)^2+o(t^2)}{t^2}$$
$$=\lim_{t\to0^+}\frac{-2t^2+1/2t^2+o(t^2)}{t^2}=\lim_{t\to0^+}\frac{-3/2t^2+o(t^2)}{t^2}=\lim_{t\to0^+}\frac{-3/2+\overset{\nearrow 0}{o(1)}}{1}=-3/2$$

⑤
$$\lim_{x\to0}\frac{1-\cos(1-\cos x)}{x^4}=$$

$$\ulcorner\cos(1-\cos x)=\cos\left(1-\left(1-\frac{x^2}{2}+\frac{x^4}{24}+o(x^4)\right)\right)=\cos\left(\frac{x^2}{2}-\frac{x^4}{24}+o(x^4)\right)$$
$$=1-\frac{\left(\frac{x^2}{2}-\frac{x^4}{24}+o(x^4)\right)^2}{2}+o\left(\left(\frac{x^2}{2}-\frac{x^4}{24}+o(x^4)\right)^2\right)=1-\frac{x^4}{8}+o(x^4)\lrcorner$$
$$\Rightarrow\lim_{x\to0}\frac{1-\cos(1-\cos x)}{x^4}=\frac{\cancel{1}-\cancel{1}+\frac{x^4}{8}+o(x^4)}{x^4}=\frac{1}{8}$$

⑥
$$\lim_{x\to0}\frac{(1+\cos x)^{(1-\cos x)}-1}{(\ln(1+x))^2}$$

$$\ln2+\ln\left(1+\frac{\cos x-1}{2}\right)$$
$$\uparrow$$

$$\ulcorner(\ln(1+x))^2=(x+o(x))^2=x^2+o(x^2)$$
$$(1+\cos x)^{(1-\cos x)}=e^{(1-\cos x)\ln(1+\cos x)}=e^{\left(\frac{x^2}{2}+o(x^2)\right)\ln(1+\cos x)}=e^{x^2\left(1/2+o(1)\right)\left(\ln2+o(1)\right)}$$
$$=e^{x^2\left(\frac{\ln2}{2}+o(1)\right)}=1+x^2\frac{\ln2}{2}+o(x^2),\ x\to0$$
$$\lim_{x\to0}\frac{(1+\cos x)^{(1-\cos x)}-1}{(\ln(1+x))^2}=\lim_{x\to0}\frac{1+x^2\frac{\ln2}{2}+o(x^2)-1}{x^2+o(x^2)}=\lim_{x\to0}\frac{\frac{x^2\ln2}{2}+o(x^2)}{x^2+o(x^2)}=\lim\frac{\frac{\ln2}{2}+\overset{\nearrow 0}{o(1)}}{1+\underset{\searrow 0}{o(1)}}=\frac{\ln2}{2}$$

Čas 10

① Domen (ne delimo nulom, arccos, arcsin između -1 i 1)

② Nule i znak

③ Parnost/Neparnost, periodičnost

④ Asimptote

1) Vertikalne
2) Horizontalne/kose

⑤ Monotonost i lokalni ekstremumi

$f'(x) \to$ nule i znak

$f'(x)>0 \to f \uparrow$ u x

$f'(x)<0 \to f \downarrow$ u x

$= 0 \to$ potencijalni ekstremumi

⑥ Konveksnost + Prevojne tačke

$f''(x)>0 \to$ konv. $f''(x)=0$

$f''(x)<0 \to$ konk.

Zadaci:

1) $f(x) = |x+2|\, e^{-\frac{1}{x}}$

1) Domen $D=(-\infty,0)\cup(0,+\infty)$, $D=\mathbb{R}\setminus\{0\}$

2) Nule i znak $f(x)\geq 0$ jer je $|x+2|\geq 0$ i $e^{-1/x}>0$

$f(x)=0 \iff x=-2$

3) Parnost/Neparnost, periodičnost

$f(1) = 3\cdot\frac{1}{e}$ $\frac{3}{e}\neq e$ nije parna

$f(-1) = 1\cdot e$ $\frac{3}{e}\neq -e$ nije neparna

4) Asimptote

1) Vertikalne

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \overbrace{|x+2|}^{2}\, \underbrace{e^{\overset{-\infty}{\left(-\overset{+\infty}{1/x}\right)}}}_{0} = 2\cdot 0 = 0$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \underbrace{|x+2|}_{2}\, \underbrace{e^{\left(-\frac{1}{x}\right)\to +\infty}}_{+\infty} = +\infty \qquad \lim_{x\to 0^-} f(x) = +\infty$$

a) kose/horizontalne

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} |x+2|\, e^{-1/x} = +\infty \quad \left(\begin{matrix}\text{Nema horizontalne,}\\ \text{proverimo kose}\end{matrix}\right)$$

$$a_1 = \lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \frac{|x+2|\, e^{-1/x}}{x} = 1$$

$$b_1 = \lim_{x\to+\infty} (f(x) - a_1 x) = \lim_{x\to+\infty} x\left(\frac{x+2}{x}\cdot e^{-1/x} - 1\right) = \lim_{x\to+\infty} x\left(\left(1+\frac{2}{x}\right)e^{-1/x} - 1\right) = \ulcorner t = 1/x \lrcorner =$$

$$= \lim_{t\to0^+} \frac{(1+2t)e^{-t} - 1}{t} = \lim_{t\to0^+} \frac{(1+2t)(1-t+o(t)) - 1}{t} = \lim_{t\to0^+} \frac{1+2t-t+o(t)-1}{t} = 1$$

Kada $x\to+\infty$ imamo kosu asimptotu $x+1$

II način

$$f(x) = a_1\cdot x + b_1 + o(1),\ x\to+\infty$$

$$= x\left(a_1 + \frac{b_1}{x} + o\left(\frac{1}{x}\right)\right),\ x\to+\infty$$

$$f(x) = x\left(\frac{x+2}{x}\cdot e^{-1/x}\right) = x\left(\left(1+\frac{2}{x}\right)\cdot e^{-1/x}\right) = x\left(\left(1+\frac{2}{x}\right)\left(1-\frac{1}{x}+o\left(\frac{1}{x}\right)\right)\right),\ x\to+\infty$$

$$f(x) = x\left(1+\frac{2}{x}-\frac{1}{x}+o\left(\frac{1}{x}\right)\right) = x+1+o(1),\ x\to+\infty$$

1) Horizontalna za $-\infty$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \underset{\to +\infty}{|x+2|}\ \overset{\to 1}{e^{-1/x}} = +\infty \quad \text{Nema!}$$

2) kose

$$a_2 = \lim_{x\to-\infty} f(x)\cdot\frac{1}{x} = \lim_{x\to-\infty} \frac{|x+2|\, e^{-\frac{1}{x}}}{x} = \lim_{x\to-\infty} \overset{\to -1}{\boxed{\frac{-x-2}{x}}}\cdot\overset{\to 1}{\boxed{e^{-1/x}}} = -1$$

$$b_2 = \lim_{x\to-\infty} (f(x) - a_2 x) = \lim_{x\to-\infty} ((-x-2)\, e^{-1/x} + x) = \lim_{x\to-\infty} x\left(\frac{-x-2}{x}\cdot e^{-1/x} + 1\right) = \ulcorner t = \frac{1}{x} \lrcorner =$$

$$= \lim_{t\to0^-} \frac{1-(1+2t)e^{-t}}{t} = \lim_{t\to0^-} \frac{(1+2t)(1-t+o(t))}{t} =$$

5. Monotonost

I $x > -2 \Rightarrow |x+2| = x+2$

$$f'(x) = ((x+2)e^{-1/x})' = e^{-1/x} + (x+2)e^{-\frac{1}{x}}\frac{1}{x^2} = e^{-\frac{1}{x}}\left(1+\frac{x+2}{x^2}\right) = \overset{>0}{\boxed{\frac{e^{-\frac{1}{x}}}{x^2}}}\underset{>0}{(x^2+x+2)}$$

$f'(x) > 0$, Za $x > -2$ $f(x)\nearrow$

II $x < -2 \Rightarrow |x+2| = -x-2$

$$f'(x) = -\frac{e^{-1/x}}{x^2}(x^2+x+2) < 0$$

Za $x < -2$ $f'(x) < 0$, tj. $f(x)\downarrow$

$$\lim_{h\to 0^+}\frac{f(-2+h)-f(-2)}{h}=\frac{e^{+1/2}}{4}\cdot 4=\sqrt{e}$$

$$\lim_{h\to 0^-}\frac{f(-2+h)-f(-2)}{h}=\frac{e^{1/2}}{4}\cdot 4=-\sqrt{e}$$

$f(-2)=0$ → lok. minimum.

⑥ Konveksnost

I $x>-2$ $\quad f''(x)=e^{-\frac{1}{x}}\left(-\frac{1}{x^2}-\frac{4}{x^3}\right)+e^{-\frac{1}{x}}\cdot\frac{1}{x^2}\left(1+\frac{1}{x}+\frac{2}{x^2}\right)$

$$=\frac{e^{-1/x}}{x^4}\left(\cancel{-x^2}-4x+\cancel{x^2}+x+2\right)=\underset{>0}{\boxed{\frac{e^{-1/x}}{x^4}}}(2-3x)$$

$f''(x)>0$ za $x\in(-2,\frac{2}{3})$ $\cup$

$f''(x)<0$ za $x\in(\frac{2}{3},+\infty)$ $\cap$

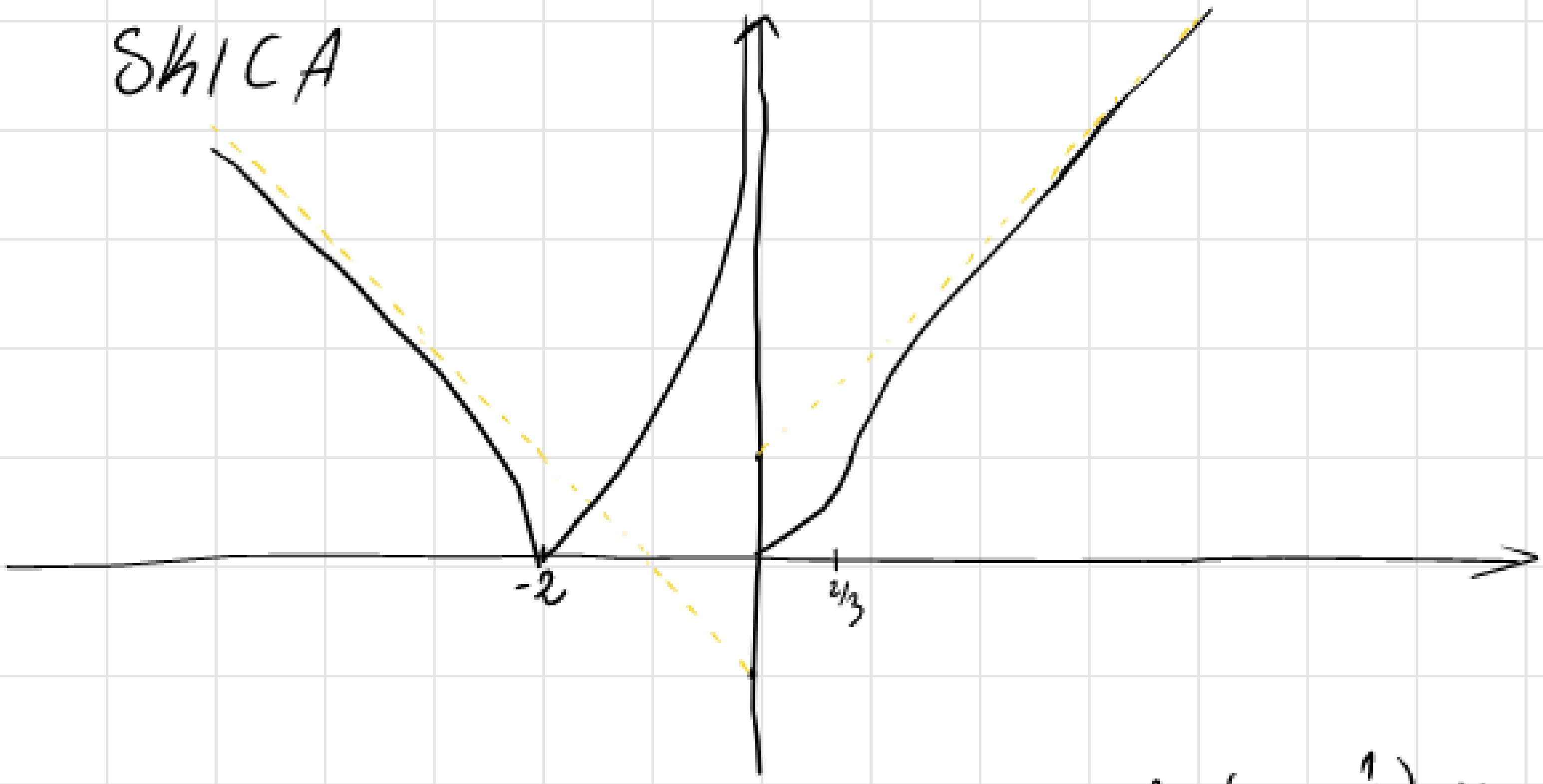
II $x<-2$

$$f''(x)=\overset{>0}{\boxed{\frac{e^{-1/x}}{x^4}}}\underset{x=3}{(3x-2)}<0 \quad \xleftarrow{x<-2}$$

f konveksna na $(-2,0)\cup(0,\frac{2}{3})$

f konkavna na $(-\infty,-2)\cup(\frac{2}{3},+\infty)$

SKICA



2) $f(x)=-x+\ln\left(\frac{|x|-1}{x}\right)$ $\begin{cases}-x+\ln\left(1-\frac{1}{x}\right), & x>1\\ -x+\ln\left(-1-\frac{1}{x}\right), & -1<x<0\end{cases}$

1) Domen: $x\neq 0$, $x\neq 1$, $x\neq -1$ $\quad \frac{|x|-1}{x}>0$

$$\left.\begin{matrix}\text{I}\quad x>0 \quad \frac{x-1}{x}>0,\ x>1\\ \text{II}\quad x<0 \quad -\frac{(x+1)}{x}<0,\ -1<x<1\end{matrix}\right\} D_f=(-1,0)\cup(1,+\infty)$$

2) Nule i znak

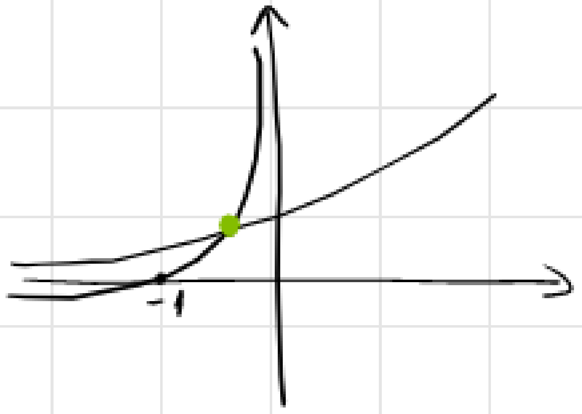
$$x>1:\left.\begin{matrix}\ln\left(1-\frac{1}{x}\right)<0\\ -x<0\end{matrix}\right\} f(x)<0$$

$-1 < x < 0 \qquad \ln(-1 - \frac{1}{x}) - x = 0$

Ne znamo da rešimo vratićemo se na ovo kasnije

$\iff \ln(-1 - \frac{1}{x}) = x / e$

$\iff -1 - \frac{1}{x} = e^x$



3. Parnost/Neparnost, periodičnost

- D nije simetričan $\Rightarrow$ nije ni parna ni neparna
- Nije periodična.

4 Asimptote

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} \Big(\overset{+1}{\boxed{-x}} + \overset{-\infty}{\boxed{\ln(\overset{0}{\boxed{-1-\frac{1}{x}}})}}\Big) = -\infty$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \Big(\overset{0^+}{\boxed{-x}} + \ln(\underset{+\infty}{\boxed{-1-\frac{1}{x}}})\Big) = +\infty$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \Big(-x + \underset{-\infty}{\boxed{\ln(\underset{0}{\boxed{1-\frac{1}{x}}})}}\Big) = -\infty$$

Kosa?

$$a = \lim_{x\to +\infty} \frac{(-x + \ln(1-\frac{1}{x}))}{x} = \lim_{x\to +\infty} \Big(-1 + \overset{\to 0}{\boxed{\frac{\ln(\overset{1}{\boxed{1-\frac{1}{x}}})}{x}}}\Big) = -1$$

$$b = \lim_{x\to +\infty} (-x + \ln(1-\frac{1}{x}) + x) = \lim_{x\to +\infty} \ln(-\frac{1}{x}) = 0$$

$\Rightarrow$ Kada $x \to +\infty$ imamo kosu asimptotu $y = -x$

⑤ Monotonost

I $\quad x > 1$

$$f'(x) = (-x + \ln(1-\frac{1}{x}))' = -1 + \frac{1}{1-\frac{1}{x}} \cdot \frac{1}{x^2} = -1 + \frac{1}{x^2 - x}$$

$$= \frac{x - x^2 + 1}{x(x-1)} = -\frac{x^2 - x - 1}{\underset{>0}{\boxed{x(x-1)}}} \qquad x_{1/2} = \frac{1 \pm \sqrt{5}}{2}$$

$x^2 - x - 1 < 0 \iff x \in (\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{3}}{2})$

$f'(x) > 0$ kada $x \in (1, \frac{1+\sqrt{3}}{2}) \nearrow$

$f'(x) < 0$ kada $x \in (1 + \frac{\sqrt{5}}{2}, +\infty) \searrow$

$\Big\}$ $\frac{1+\sqrt{3}}{2}$ je lokalni maximum

II $\quad x \in (-1, 0)$

$$f'(x) = -1 + \frac{1}{-1-\frac{1}{x}} \cdot \frac{1}{x^2} = -\Big(1 + \frac{1}{x^2+x}\Big) = -1 \cdot \Big(\frac{\overset{>0}{\boxed{x^2+x+1}}}{x^2+x}\Big) = \underset{<0}{\boxed{-}} \frac{\overset{>0}{\boxed{x^2+x+1}}}{\underset{<0}{\boxed{x}}\underset{>0}{\boxed{(x+1)}}} > 0$$

$f \nearrow$ na $(-1, 0)$

⑥ Konveksnost

I $x>1$ $f''(x) = \left(-1+\frac{1}{x-1}-\frac{1}{x}\right)' = \frac{1}{(x-1)^2} + \frac{1}{x^2} = \frac{-x^2+x^2-2x+1}{x^2(x-1)^2} = \frac{-2x+1}{x^2(x-1)^2}$

$x>1$ $-2x+1<-1$ tj. $\frac{-2x+1}{(x(x+1))^2}<0$ $\cap$

$f(-\frac{1}{2}) = -\frac{1}{2}$

II $x\in(-1,0)$

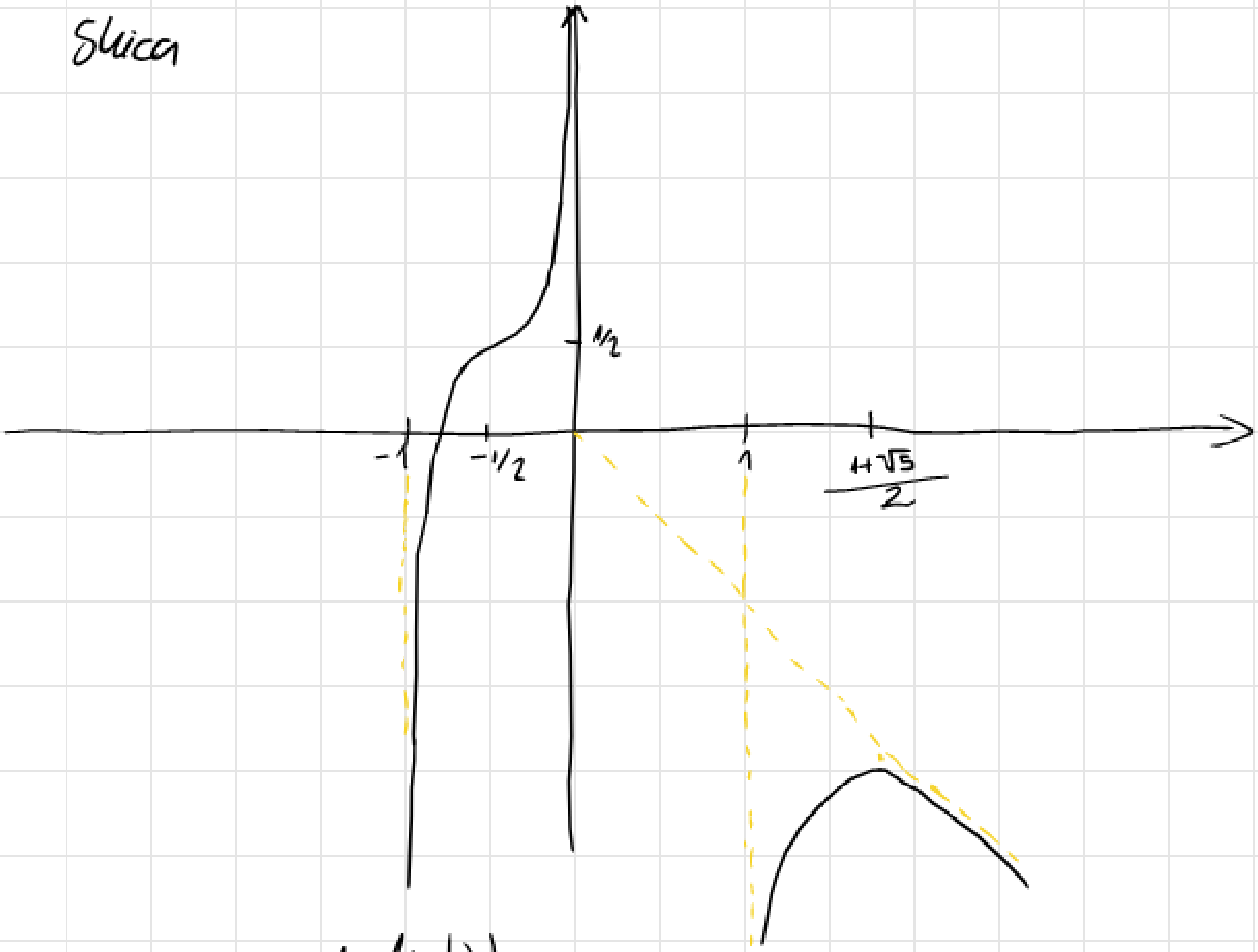
$f''(x) = \frac{2x+1}{x^2(x+1)^2}$ $2x+1>0$ $x>-\frac{1}{2}$

$f''(x)$ $\cup$ $x\in(-\frac{1}{2},0)$

$f''(x)$ $\cap$ $x\in(-1,-\frac{1}{2})$

$-\frac{1}{2}$ prevojna

Slika



③ $f(x) = \frac{1+\ln|x|}{x(1-\ln|x|)}$

① Domen $x\in\mathbb{R}\setminus\{0,-e,e\}$

② Nule i znak

$f(x)=0 \iff 1+\ln|x|=0 \iff x=\pm 1/e$

	$-\infty$ $-e$	$-e$ $-1/e$	$-1/e$ 0	0 $1/e$	$1/e$ e	e $+\infty$
$1+\ln\|x\|$	+	+	−	−	+	+
$1-\ln\|x\|$	−	+	+	+	+	−
x	−	−	−	+	+	+
$f(x)$	+	−	+	−	+	−

③ Parnost/Neparnost periodičnost → nije

$f(-x) = \frac{1+\ln|x|}{-x(1-\ln|x|)} = -f(x)$ ⟹ f neparna Ispitujemo samo $(0,e)\cup(e,+\infty)$

4. Asimptote

$$\lim_{x\to0^+} f(x) = \lim_{x\to0^+}\frac{1+\overset{-\infty}{\ln x}}{x(1-\ln x)} = \lim_{x\to0^+}\frac{1}{x}\cdot\frac{1+\ln x}{1-\ln x} = \lim_{x\to0^+}\overset{+\infty}{\frac{1}{x}}\left(-1+\overset{0}{\frac{2}{\underset{-\infty}{1-\ln x}}}\right) = -\infty$$