REMARK ON THE CONVERGENCE OF A SEQUENCE*

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In the present note we give a considerably simplified proof of a theorem proved in [1]. Also an unnecessary assumption from [1] is omitted.

Let $E$ be a complete metric space, $f_n: E^k \to E$ a sequence of mappings of the product space $E^k$ into $E$ such that

\[ d(f_n(u_1, u_2, \ldots, u_k), f_n(u_2, u_3, \ldots, u_{k+1})) < q_1 d(u_1, u_2) + q_2 d(u_2, u_3) + \cdots + q_k d(u_k, u_{k+1}), \]

for every $u_1, u_2, \ldots, u_{k+1} \in E$, where $q_1, q_2, \ldots, q_k$ are non-negative fixed numbers such that $q_1 + q_2 + \cdots + q_k < q < 1$.

Let

\[ d(f_{n+1}(u_1, u_2, \ldots, u_k), f_n(u_1, u_2, \ldots, u_k)) < a_n \quad (n = 1, 2, \ldots), \]

where $a_n$, $(n = 1, 2, \ldots)$, are positive terms of a convergent series. (In [1] it has been supposed that $\lim \inf a_{n+1} = 1$.) It is easily seen that the sequence $f_n$, $n = 1, 2, \ldots$ converges uniformly to a function $f: E^k \to E$.

We prove namely the following:

Theorem. Let

\[ x_{n+k} = f_n(x_n, x_{n+1}, \ldots, x_{n+k-1}), \quad (n = 1, 2, \ldots), \]

where the elements $x_1, x_2, \ldots, x_k$ are arbitrarily chosen. Then, if the conditions (1) and (2) are satisfied:

1. The sequence $x_n$ converges in $E$.
2. The equation

\[ x = f(x, x, \ldots, x) \]

has a unique solution $x = \lim_{n \to \infty} x_n$.

Proof. 1. Putting $\Delta_n = d(x_n, x_{n+1})$, then by conditions (1) and (2) we get the following system of inequalities

\[ \Delta_{n+k+i} < a_{n+i} + q_1 \Delta_{n+i} + q_2 \Delta_{n+i+1} + \cdots + q_k \Delta_{n+k-1+i} \quad (i = 0, 1, 2, \ldots, s). \]

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Adding together the inequalities (3) and putting \( \sigma_{n+k+s}^{n+k} = \sum_{v=0}^{s} \Delta_{n+k+v} \), we easily find that
\[
\sigma_{n+k+s}^{n+k} \leq \sum_{v=0}^{n+k+s} a_v + q \sigma_{n+k+s}^{n+k} + q (\Delta_n + \Delta_{n+1} + \cdots + \Delta_{n+k-1}).
\]
Therefore,
\[
\sigma_{n+k+s}^{n+k} \leq \frac{1}{1-q} \sum_{v=0}^{n+k+s} a_v + \frac{q}{1-q} (\Delta_n + \Delta_{n+1} + \cdots + \Delta_{n+k-1}).
\]
From
\[
\limsup_{n \to \infty} \Delta_{n+k} < q_1 \limsup_{n \to \infty} \Delta_n + \cdots + q_k \limsup_{n \to \infty} \Delta_{n+k-1},
\]
i.e.,
\[
\limsup_{n \to \infty} \Delta_n < q \limsup_{n \to \infty} \Delta_n,
\]
we find that \( \Delta_n \to 0 \), as \( n \to \infty \). Letting \( n \to \infty \) in (4) we obtain
\[
d (x_{n+k}, x_{n+k+s}) < \sigma_{n+k+s}^{n+k} \to 0.
\]
Being \( E \) complete, the sequence \( x_n \) converges, i.e. \( \lim x_n = x_0 \).

2. Applying triangle inequality, we get
\[
d (f(u_1, u_2, \ldots, u_k), f(u_2, u_3, \ldots, u_{k+1})) < d (f(u_1, u_2, \ldots, u_k),
\]
\[
+ d (f_n (u_1, u_2, \ldots, u_k), f (u_2, u_3, \ldots, u_{k+1}))
\]
\[
< 2 \sum_{v=n}^{\infty} a_v + q_1 d (u_1, u_2) + \cdots + q_k d (u_k, u_{k+1}),
\]
wherefrom, as \( n \to \infty \) we get
\[
d (f(u_1, \ldots, u_k), f(u_2, \ldots, u_{k+1})) < q_1 d (u_1, u_2) + \cdots + q_k d (u_k, u_{k+1}).
\]
Now, we have
\[
d (f_n (x_n, x_{n+1}, \ldots, x_{n+k-1}), f (x_0, x_0, \ldots, x_0)) = d (x_{n+k}, f (x_0, x_0, \ldots, x_0))
\]
\[
< d (f (x_n, x_{n+1}, \ldots, x_{n+k-1}), f (x_0, x_0, \ldots, x_0)) + \sum_{v=n}^{\infty} a_v \to 0
\]
as \( n \to \infty \), what implies that \( x_0 = f (x_0, x_0, \ldots, x_0) \).

The uniqueness of the solution follows, for example, from the Banach contraction theorem applied to
\[
\overline{f} (u) = f (u, u, \ldots, u), \quad (u \in E)
\]
since
\[
d (\overline{f} (u), \overline{f} (v)) < q d (u, v) \quad (u, v \in E).
\]

REFERENCE