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REMARK ON THE CONVERGENCE OF A SEQUENCE*

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In the present note we give a considerably simplified proof of a theorem proved in [1]. Also an unnecessary assumption from [1] is omitted.

Let E be a complete metric space, $f_n: E^k \to E$ a sequence of mappings of the product space E^k into E such that

$$(1) \qquad d(f_n(u_1, u_2, \ldots, u_k), f_n(u_2, u_3, \ldots, u_{k+1})) \leqslant q_1 d(u_1, u_2) + q_2 d(u_2, u_3) + \cdots + q_k d(u_k, u_{k+1}),$$

for every $u_1, u_2, \ldots, u_{k+1} \in E$, where $q_1, q_2, \ldots, q_k$ are non-negative fixed numbers such that $q_1 + q_2 + \cdots + q_k \leqslant q < 1$.

Let

$$(2) \qquad d(f_{n+1}(u_1, u_2, \ldots, u_k), f_n(u_1, u_2, \ldots, u_k)) \leqslant a_n \qquad (n = 1, 2, \ldots),$$

where a_n, $(n = 1, 2, \ldots,)$ are positive terms of a convergent series. $\left(\text{In [1] it has been supposed that } \liminf_{n\to\infty} \frac{a_{n+1}}{a_n} = 1\right)$. It is easily seen that the sequence f_n, $n = 1, 2, \ldots$ converges uniformly to a function $f: E^k \to E$.

We prove namely the following:

Theorem. Let

$$x_{n+k} = f_n(x_n, x_{n+1}, \ldots, x_{n+k-1}), \quad (n = 1, 2, \ldots),$$

where the elements $x_1, x_2, \ldots, x_k$ are arbitrarily chosen. Then, if the conditions (1) *and* (2) *are satisfied:*

1. *The sequence x_n converges in E.*
2. *The equation*

$$x = f(x, x, \ldots, x)$$

has a unique solution $x = \lim_{n\to\infty} x_n$.

Proof. 1. Putting $\Delta_n = d(x_n, x_{n+1})$, then by conditions (1) and (2) we get the following system of inequalities

$$(3) \qquad \Delta_{n+k+i} \leqslant a_{n+i} + q_1 \Delta_{n+i} + q_2 \Delta_{n+1+i} + \cdots + q_k \Delta_{n+k-1+i} \qquad (i = 0, 1, 2, \ldots, s).$$

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Adding together the inequalities (3) and putting $\sigma_{n+k+s}^{n+k}=\sum_{\nu=0}^{s}\Delta_{n+k+\nu}$, we easily find that

$$\sigma_{n+k+s}^{n+k}\leqslant\sum_{\nu=n}^{n+s}a_\nu+q\,\sigma_{n+k+s}^{n+k}+q\,(\Delta_n+\Delta_{n+1}+\cdots+\Delta_{n+k-1}).$$

Therefore,

(4) $$\sigma_{n+k+s}^{n+k}\leqslant\frac{1}{1-q}\sum_{\nu=n}^{n+s}a_\nu+\frac{q}{1-q}(\Delta_n+\Delta_{n+1}+\cdots+\Delta_{n+k-1}).$$

From

$$\limsup_{n\to\infty}\Delta_{n+k}\leqslant q_1\limsup_{n\to\infty}\Delta_n+\cdots+q_k\limsup_{n\to\infty}\Delta_{n+k-1},$$

i. e.,

$$\limsup_{n\to\infty}\Delta_n\leqslant q\limsup_{n\to\infty}\Delta_n,$$

we find that $\Delta_n\to 0$, as $n\to\infty$. Letting $n\to\infty$ in (4) we obtain

$$d\,(x_{n+k},\ x_{n+k+s})\leqslant\sigma_{n+k+s}^{n+k}\to 0.$$

Being E complete, the sequence x_n converges, i. e. $\lim x_n=x_0$.

2. Applying triangle inequality, we get

$$d\,(f(u_1,u_2,\ldots,u_k),\ f(u_2,u_3,\ldots,u_{k+1}))\leqslant d\,(f(u_1,u_2,\ldots,u_k),$$
$$f_n(u_1,u_2,\ldots,u_k))+d\,(f_n(u_1,u_2,\ldots,u_k),\ f_n(u_2,u_3,\ldots,u_{k+1}))$$
$$+\,d\,(f_n(u_2,u_3,\ldots,u_{k+1}),\ f(u_2,u_3,\ldots,u_{k+1}))$$
$$\leqslant 2\sum_{\nu=n}^{\infty}a_\nu+q_1\,d\,(u_1,u_2)+\cdots+q_k\,d\,(u_k,u_{k+1}),$$

wherefrom, as $n\to\infty$ we get

$$d\,(f(u_1,\ldots,u_k),\ f(u_2,\ldots,u_{k+1}))\leqslant q_1\,d\,(u_1,u_2)+\cdots+q_k\,d\,(u_k,u_{k+1}).$$

Now, we have

$$d\,(f_n(x_n,x_{n+1},\ldots,x_{n+k-1}),\ f(x_0,x_0,\ldots,x_0))=d\,(x_{n+k},f(x_0,x_0,\ldots,x_0))$$
$$\leqslant d\,(f(x_n,x_{n+1},\ldots,x_{n+k-1}),\ f(x_0,x_0,\ldots,x_0))+\sum_{\nu=n}^{\infty}a_\nu\to 0$$

as $n\to\infty$, what implies that $x_0=f(x_0,x_0,\ldots,x_0)$.

The uniqueness of the solution follows, for example, from the Banach contraction theorem applied to

$$\bar f(u)=f(u,u,\ldots,u),\qquad(u\in E)$$

since

$$d\,(\bar f(u),\ \bar f(v))\leqslant q\,d\,(u,v)\qquad(u,\ v\in E).$$

REFERENCE

[1] S. B. Prešić: *Sur la convergence des suites*, Comptes rendus de l'Académie des sciences de Paris, t. 260, 1965.