ON THE EMBEDDING OF $\Omega$-ALGEBRAS IN GROUPOIDS

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Summary. It has been proved that any $\Omega$-algebra can be embedded both in a semigroup [1], [3], and in a so called entropic groupoid [2].

This paper gives a necessary and sufficient condition for embedding any $\Omega$-algebra in some groupoid $\mathcal{G} = (G, \ast)$ satisfying the set of laws $\Sigma(\ast)$. The condition is:

There is a term $\xi(x, y; \ast)$ formed of two variables $x, y$ and the operation symbol $\ast$ such that the operation $\circ$ defined by

$$(C) \quad xy \circ \overset{\text{def}}{=} \xi(x, y; \ast)$$

does not satisfy any algebraic law (except the law $x = x$), while the operation $\ast$ satisfies the laws in $\Sigma(\ast)$.

Some new examples of groupoids which satisfy the condition (C) are presented:

$1^o$ Each $\Omega$-algebra can be embedded in a commutative groupoid.

$2^o$ Each $\Omega$-algebra can be embedded in a groupoid satisfying the law of the type

$$\Pi x_1 \cdots x_n = \Pi x_{p_1} \cdots x_{p_n},$$

where at both sides of the equality the arrangement of the operation symbols is the same, and the permutation $\begin{pmatrix} 1 & 2 & \cdots & n \\ p_1 & p_2 & \cdots & p_n \end{pmatrix}$ has at least one fixed point.

For example, the laws of this type are

$$xy \ast z \ast = zy \ast x \ast, \quad xy \ast u \ast vw \ast \ast = wy \ast y \ast ux \ast \ast.$$ 

In the second part of the paper it is shown that the condition (C) can be extended so that the main theorems 1, 2 hold for the laws $\Sigma(\ast)$, i.e. for variety $V_O(\Sigma)$, where $O$ is a set of some operation symbols. For example, any $\Omega$-algebra can be embedded in an entropic algebra, i.e. in an algebra satisfying:

$$(x_{11}x_{12} \cdots x_{1n}f \cdots x_{n1}x_{n2} \cdots x_{nn}f) \overset{\text{def}}{=} (x_{11}x_{21} \cdots x_{n1}f \cdots x_{1n}x_{2n} \cdots x_{nn}f) f$$

$1^o$ In other words, $\mathcal{G}$ belongs to the variety $V_\ast(\Sigma)$, [1].
1. The main result of the paper is the following

**Theorem 1.** If \( Q \) is an arbitrary \( \Omega \)-algebra and \( \Sigma (\ast) \) a set of laws\(^1\) satisfying the condition (\( C \)), then there exists a groupoid \( \mathcal{G} = (G, \ast) \) satisfying the laws \( \Sigma (\ast) \), with the properties:

(i) \( Q \) is a subset of \( G \);

(ii) If \( \omega \in \Omega \) is some \( n \)-ary operation, then there exists an \( \overline{\omega} \in G \) and a term \( t_{\omega}(x_1, \ldots, x_n, \omega; \circ) \) formed from variables \( x_1, \ldots, x_n \), the constant symbol \( \overline{\omega} \) and the operation symbol \( \circ \) such that:

\[
\omega = \overline{\omega}, \quad \text{if} \quad \omega \in \Omega (\circ)
\]

\[
x_1 x_2 \cdots x_n \omega = t_{\omega}(x_1, \ldots, x_n, \overline{\omega}; \circ), \quad \text{if} \quad \omega \in \Omega (n), \quad n = 1, 2, \ldots
\]

where \( xy^\circ \overset{\text{def}}{=} \xi(x, y; \ast) \). The existence of \( \xi(x, y; \ast) \) is guaranteed by (\( C \)).

For the proof we need the following definitions:

I. Let \( G \) be a minimal set satisfying the conditions:

The set \( X \overset{\text{def}}{=} Q \cup \Omega \) is a subset of \( G \),

If \( u, v \in G \), then \( uv \ast \in G \).

Let \( \mathcal{G} = (G, \ast) \) be a groupoid determined by the set \( G \) and by the operation defined in the natural way, i.e. \( \mathcal{G} \) is the \( \ast \)-word algebra on \( X \).

II. Let \( G_0 \) be a minimal subset of \( G \) such that:

\( X \) is a subset of \( G_0 \),

If \( u, v \in G_0 \), then \( uv \circ \in G_0 \), where \( \circ \) is defined by the term \( \xi(x, y; \ast) \) (of condition (\( C \))):

**Def (\( \circ \))**

\[
xy^\circ \overset{\text{def}}{=} \xi(x, y; \ast)
\]

III. For each \( \omega \in \Omega (n) \) \((n = 0, 1, \ldots)\) we define an operation \( \omega \) on \( G \):

**Def (\( \Omega \))**

\[
\omega = \omega, \quad \text{if} \quad n = 0,
\]

\[
x_1 \cdots x_n \omega = t_{\omega}(x_1, \ldots, x_n, \omega; \circ), \quad \text{if} \quad n = 1, 2, \ldots,
\]

where \( t_{\omega}(x_1, \ldots, x_n, \omega; \circ) \) is a term formed from variables \( x_1, \ldots, x_n \), the constant \( \omega \) and the operation symbol \( \circ \). These terms can be chosen arbitrarily, in particular, they can be chosen with the operation symbols grouped on the right [2]. The operations defined in that way form the set

\[
\Omega \overset{\text{def}}{=} \{ \omega \mid \omega \in \Omega \}.
\]

IV. Finally, let \( G_\Omega \) be a minimal subset of \( G_0 \) such that:

\( Q \cup \Omega (0) \) is a subset of \( G_\Omega \),

If \( u_1, \ldots, u_n \in G_\Omega \), and \( \omega \in \Omega (n) \) \((n = 1, 2, \ldots)\), then \( u_1 \cdots u_n \omega \in G_\Omega \).

Hence, the elements of \( G_\Omega \) are those terms of \( G \) which can be represented by operations from \( \Omega \).

\(^{1}\) We assume that the laws are consistent, i.e. that they do not imply \( x = y \), where \( x, y \) are different variables.
V. Let $\sim_z$, $\sim_Q$, $\sim_o$, $\sim_\Omega$ be the minimal congruences generated by

$$\Sigma(*)$$, \quad Tab Q(\Omega), \quad Def(o), \quad Def(\Omega),$$

that is:

$$u \sim_z v \iff \Sigma(*) \vdash u = v,$$

$$u \sim_Q v \iff \text{Tab } Q(\Omega) \vdash u = v, \quad (u, v \in G)$$

$$u \sim_o v \iff \text{Def}(o) \vdash u = v$$

$$u \sim_\Omega v \iff \text{Def}(\Omega) \vdash u = v.$$

The symbol $\vdash$ denotes the logical deduction [4], for example $\Sigma(*) \vdash u = v$ means that the formula $u = v$ can be derived from the laws $\Sigma(*)$ and the equality axioms. Tab $Q(\Omega)$ is the so called positive diagram [4] of the algebra $Q$, i.e., the set of all equalities of the form

$$a_1 \cdots a_n \omega = a \quad (a_1, \ldots a_n, a \in Q, \omega \in \Omega(n), \; n = 1, 2, \ldots)$$

which hold in $Q$. Tab $Q(\Omega)$ is the corresponding set of formulas obtained by exchanging each $\omega \in \Omega$ by $\omega \in \Omega$.

VI. The relation $\sim$ is the minimal congruence of the set $G$ generated by

$$\sim_z, \sim_Q, \sim_o, \sim_\Omega.$$

**Lemma.** Let $\rho$ be one of the relations $\sim_z$, $\sim_Q$, $\sim_o$, $\sim_\Omega$ and let $\sigma$ be one of the relations $\sim_z$, $\sim_o$, $\sim_\Omega$. Then

$$(u \in G_\Omega \land u \rho v) \Rightarrow (\exists v' \in G_\Omega) v \sigma v'.$$

**Proof.** We distinguish four cases: $u \sim_z v$, $u \sim_Q v$, $u \sim_o v$, $u \sim_\Omega v$.

Ad1. In this case the term $v'$ is just $u$, since by condition (O), each term can be uniquely represented by $o$ (if such a representation exists), and hence uniquely represented by operations from $\Omega$ (if the representation exists).

Ad2. Let $u = u(\cdots a_1 \cdots a_n \omega \cdots), a_1, \ldots a_n \in Q$. If the subterm of $u$: $a_1 \cdots a_n \omega$ is replaced by $a$, under the condition that (in $Q$) $a_1 \cdots a_n \omega = a$, the resulting term is again in $G_\Omega$. Similarly, in the case $u (\cdots a \cdots)$, i.e., $a$ is a subterm of $u$, we conclude that $u(\cdots a_1 \cdots a_n \omega \cdots) \in G_\Omega$, if $a_1 \cdots a_n \omega = a$ holds in $Q$. Since $\sim_Q$ can be defined in a finite number of such replacements, we have

$$u \in G_\Omega \land u \sim_Q v \Rightarrow v \in G_\Omega.$$

In cases 3 and 4, it immediately follows that $v'$ is uniquely determined and equal to $u$. Hence:

(1) *In the cases $u \sim_z v$, $u \sim_o v$, $u \sim_\Omega v$, if $u \in D_\Omega$, then the uniquely determined $u'$ is just $u$.

**Proof of Theorem 1.** We first prove:

(2) $\bar{x} = \bar{y} \Rightarrow x = y \quad (x, y \in Q)$,

where $\bar{x}, \bar{y}$ are the equivalence classes of $x, y$ with respect to $\sim$. 
Suppose that $\bar{x} = \bar{y}$, i.e. $x \sim y$. Then there exists a natural number $k$ and elements
\[ u_1, u_2, \ldots, u_k; \quad u_i = x, \quad u_k = y \quad (u_i \in G) \]
such that for each $i = 1, \ldots, k - 1$:
\[ u_i \sim u_{i+1} \text{ or } u_i \sim u_{i+1} \text{ or } u_i \sim u_{i+1} \text{ or } u_i \sim u_{i+1}. \]

Let $\text{Int}$ be an interpretation, i.e. a mapping such that

(i) If $x \in Q$, then $\text{Int}(x) \overset{\text{def}}{=} x$,
(ii) If $\omega \in \Omega(0)$, then $\text{Int}(\omega) \overset{\text{def}}{=} \omega$,
(iii) If $\omega \in \Omega(n) \ (n = 1, 2, \ldots)$ and $t_1, \ldots, t_n \in D_\Omega$, then
\[ \text{Int}(t_1 \cdots t_n \omega) \overset{\text{def}}{=} \text{Int}(t_1) \cdots \text{Int}(t_n) \omega \]
(iv) If $t \in D$ and there exists a $t' \in D_\Omega$ such that
\[ \Sigma(\ast), \quad \text{Def}(\circ), \quad \text{Def}(\Omega) \vdash t = t' \]
then $\text{Int}(t) \overset{\text{def}}{=} \text{Int}(t')$.

The mapping $\text{Int}$ is well defined. The conditions (i), (ii), (iii) represent the usual definition of homomorphism. The soundness of part (iv) follows from (1).

The mapping $\text{Int}$ carries the sequence $u_1, \ldots, u_k$ into the sequence
\[ \text{Int}(u_1), \text{Int}(u_2), \ldots, \text{Int}(u_k) \]
of elements of $\Omega$-algebra $Q$.

From the definition of the mapping $\text{Int}$ we have:

If one of the conditions $u \sim u$, $u \sim u$, $u \sim u$, $u \sim u$ is satisfied, then the equality $\text{Int}(u) = \text{Int}(v)$ holds in $Q$.

Hence:
\[ x = \text{Int}(u_1) = \text{Int}(u_2) = \cdots = \text{Int}(u_k) = y \]
implying that the equality $x = y$ holds in $Q$, which proves (2).

In order to complete the proof of the theorem, we introduce the quotient groupoid $\bar{G} = (\bar{G}, \ast)$, where
\[ \bar{G} \overset{\text{def}}{=} \{ \bar{x} \mid x \in G \}, \quad \bar{x} \bar{y} \ast \overset{\text{def}}{=} \bar{xy} \ast. \]
The groupoid $\bar{G}$ satisfies the same laws as $G$.

Further, let
\[ \bar{Q} \overset{\text{def}}{=} \{ \bar{x} \mid x \in Q \}. \]

For each $\omega \in \Omega(n)$ we define an operation $\bar{\omega}$ in $\bar{Q}$ in the following way
\[ \bar{\omega} \overset{\text{def}}{=} \bar{\omega}, \quad \text{if} \quad \omega \in \Omega(0) \]
\[ \bar{x}_1 \cdots \bar{x}_n \omega \overset{\text{def}}{=} \bar{x}_1 \cdots \bar{x}_n \omega, \quad \text{if} \quad \omega \in \Omega(n) \quad (n = 1, 2, \ldots). \]
It is clear that $\overline{Q}$ is an $\Omega$-algebra ($\omega$ corresponds to $\omega$). From the first part of the proof it follows that $Q$ and $\overline{Q}$ are isomorphic algebras (an isomorphism being $f: x \rightarrow \overline{x}$). This completes the proof of the theorem.

2. We give some examples of the laws $\Sigma(\ast)$ satisfying the condition $(C)$.

I. Already known examples of the laws $\Sigma(\ast)$ are associative and entropic laws ([1], [3], [2]). The simplest terms $\xi(x, y; \ast)$ in the first case are

$$ax \ast y \ast, \ xy \ast a \ast$$

and in the second one

$$xa \ast ay \ast, \ xx \ast xy \ast, \ xy \ast yy \ast.$$ 

II. The commutative law also satisfies the condition $(C)$. One example of the term $\xi(x, y; \ast)$ is: $xx \ast xy \ast$. This is the well known term which is used to define the ordered pair.\(^1\)

III. If $\Sigma(\ast)$ is a law of the form

$$\Pi x_1 \cdots x_n = \Pi x_{p_1} \cdots x_{p_n}$$

where at both sides of the equality are terms with the same arrangement of operation symbols, and the permutation $p = \left(1 \cdots n\right) = \left(p_1 \cdots p_n\right)$ has at least one fixed point, for example $p(i) = i$, then one convenient term is

$$\Pi x \cdots xyx \cdots x \quad (y \text{ is at the } i\text{-th place})$$

In the case of laws:

$$xy \ast z \ast = yz \ast x \ast, \ \ xy \ast u \ast vv \ast = wy \ast v \ast ux \ast$$

such terms are: $xy \ast x \ast$, $xy \ast x \ast xx \ast$ respectively, for they stay unchanged after applying the corresponding laws [2].

IV. An example of laws of the previous type is

$$\Pi x_1 \cdots x_n = \Pi x_{p_1} \cdots x_{p_n}$$

under the condition that the permutation $p$ has at least two fixed points, say $p(i) = i$, $p(j) = j$. The convenient term $\xi(x, y; \ast)$ can be formed as in III, as well as in different way. Namely

$$\Pi a \cdots axa \cdots aya \cdots a \quad (a \text{ is a constant; } x \text{ and } y \text{ are at the } i\text{-th and } j\text{-th place})$$

satisfies also the condition $(C)$.

Remark. If the term $\xi(x, y; \ast)$ is such that it depends on the constant $a$, i.e. the term of the form $\xi(x, y; a; \ast)$, then $\text{Def}(\omega)$ becomes:

Definition ($\omega_a$)

$$xy^{\omega_a} \overset{\text{def}}{=} \xi(x, y, a; \ast)$$

and the operation $\omega$ can be represented by some term $t(x_1, \ldots, x_n; \omega_a)$ i.e. it is not necessary to use a new constant, since $\omega_a$ already depends on $\omega$. This situation occurs in the case of associative and entropic laws (if for $\xi(x, y; \ast)$ the term $xa \ast ay \ast$ is chosen).

\(^{1)}\) $(x, y) = \{\{x, x\}, \{x, y\}\} = \{\{x\}, \{x, y\}\}$. 

\(\text{def}\)
3. The converse of the Theorem 1. is given by the following

Theorem 2. If \( \Sigma(*) \) are algebraic laws such that for each \( \Omega \)-algebra there exists a groupoid \( (G, *) \) satisfying the laws \( \Sigma(*) \), and \( Q \) is isomorphically embedded in \( (G, *) \), then there exists a term \( \xi(x, y; *) \) such difficult the operation \( * \) defined by \( \text{Def}(\circ) \) does not satisfy any algebraic law.

Proof. If each \( \Omega \)-algebra can be embedded in some groupoid satisfying \( \Sigma(*) \), then the same holds for \( \circ \)-word algebra \( W \) generated by some set \( X \) and the operation symbol \( \circ \) (of arity two). That is, there exists a groupoid \( (G, *) \) and a term \( \xi(x, y; *) \) such that

\[ xy \circ \equiv \xi(x, y; *). \]

As \( W \) does not satisfy any algebraic law [1], the term \( \xi(x, y; *) \) is the required term.

4. By analyzing the proofs of the previous theorems it is not hard to see that the assumption that \( \Sigma(*) \) are the groupoid laws is not essential. Namely, if \( \Sigma(\mathcal{O}) \) are the laws with respect to the operators of some set \( \mathcal{O} \), but such that:

There exists a term \( \xi(x, y; o_1, \ldots, o_n) \), \( o_i \in \mathcal{O} \) such that the operation \( \circ \)

(of arity two) defined by

\[ \xi(x, y; o_1, \ldots, o_n) \]

does not satisfy any algebraic law,

then Theorems 1 and 2 can be extended to the statements about the embeddine of each \( \Omega \)-algebra in some \( \mathcal{O} \)-algebra satisfying \( \Sigma(\mathcal{O}) \), i.e. belonging to thg variety \( V_{\mathcal{O}}(\Sigma) \). For example, one primitive class \( \Sigma(\mathcal{O}) \) satisfying \( (\xi') \) is the class \( \Sigma(\{f\}) \) with the law:

\[ (x_{11}x_{12} \cdots x_{1n}f \cdots x_{nm}f) = (x_{11}x_{21} \cdots x_{n1}f \cdots x_{nm}f). \]

The term of \( n \) variables satisfying \( (\xi') \) is:

\[ (x_{11}a \cdots af \cdots aa \cdots x_{nm}f). \]

which can easily be reduced to a term of the form \( \xi(x, y, a, f) \). The other class \( \Sigma(\mathcal{O}) \) is \( \Sigma(\{f, g\}) \) (of arity one, \( g \) of arity two) with the laws:

\[ xxg = xg, \quad xyg = yg \]

The convenient term \( \xi(x, y; f, g) \) is: \( xfxg). \) If \( f, g \) are the set operator \( \{\} \) then the term becomes \( \{\{x\}, \{x, y\}\}. \)

REFERENCES


[2] Radojčić, M. D., On the embedding of universal algebras in groupoids holding the law \( xy \circ zu = xz \circ yu \circ \), Mat. vesnik, 5 (1968), 353–356.
