GENERALIZING LOGIC PROGRAMMING TO ARBITRARY SETS OF CLAUSES

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Abstract. In this paper, which is a brief version of [3], we state how one can extend Logic Programming to any set of clauses.

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The basic part of Logic Programming, particularly Prolog, in fact deals with the following two inference rules:

1. $\mathcal{F}, p \vdash p$
2. $\mathcal{F}, p \lor \neg q_1 \lor \ldots \lor \neg q_k \vdash p \leftarrow \mathcal{F} \vdash q_1, \ldots, q_k$

(where $\mathcal{F}$ is a set of (positive) Horn formulas and $p$ is any atom, i.e. a propositional letter)

Indeed, the informal meaning of rule (1) is:

An atom $p$ is a consequence of a set of clauses if $p$ is an element of that set.

Similarly for rule (2) we have this meaning:

An atom $p$ is a consequence of a set $\mathcal{F}, p \lor \neg q_1 \lor \ldots \lor \neg q_k$ (i.e. of the set $\mathcal{F}, q_1 \land \ldots \land q_k \Rightarrow p$), if $q_1, \ldots, q_k$ are consequences of the set $\mathcal{F}$.

In the sequel we use the following facts from mathematical logic (see [2]):

3. The notion of formal proof in the case of propositional logic (assuming we have chosen some tautologies as axioms, and that modus ponens is the only inference rule).
(4) **The Deduction theorem**: \( \mathcal{F}, A \vdash B \iff \mathcal{F} \vdash A \Rightarrow B \) where \( \mathcal{F} \) is a set of propositional formulas and \( A, B \) are some such formulas.

(5) **Completeness Theorem**: Any propositional formula is a logical theorem if and only if it is a tautology.

We also use the symbols \( \bot, \top \) which can be introduced by the following definitions

\[ \bot \text{ stands for } a \land \neg a; \quad \top \text{ stands for } a \lor \neg a \]

where \( a \) is an atom (chosen arbitrarily). Further, let \( \mathcal{F} \) be any set of propositional formulas and \( \psi \) a formula or one of the symbols \( \bot, \top \). Then a **sequent** is any expression of the form \( \mathcal{F} \vdash \psi \), with the meaning:

\( \psi \) is a logical consequence of \( \mathcal{F} \)

**Lemma 1.** Let \( \mathcal{F} \) be any set of propositional formulas not containing the atom \( p \), and let \( \phi_1(p), \phi_2(p), \ldots \) be propositional formulas containing \( p \). Then we have the following equivalences

(6) (i) \( \mathcal{F}, \phi_1(p), \phi_2(p), \ldots \vdash p \iff \mathcal{F}, \phi_1(\bot), \phi_2(\bot), \ldots \vdash \bot \)

(ii) \( \mathcal{F}, \phi_1(p), \phi_2(p), \ldots \vdash \neg p \iff \mathcal{F}, \phi_1(\top), \phi_2(\top), \ldots \vdash \bot \)

**Proof.** First we give proof of the part of (i). Then, we have the following 'implication-chain':

\( \mathcal{F}, \phi_1(p), \phi_2(p), \ldots \vdash p \)

\[ \longrightarrow \text{For some formulas } f_1, \ldots, f_r \text{ of } \mathcal{F} \text{ and some formulas } \phi_{i_1}(p), \ldots, \phi_{i_s}(p) \]
we have: \( f_1, \ldots, f_r, \phi_{i_1}(p), \ldots, \phi_{i_s}(p) \vdash p \)
(Finiteness of the propositional proof)

\[ \longrightarrow \vdash f_1 \Rightarrow \ldots \Rightarrow f_r \Rightarrow \phi_{i_1}(p) \Rightarrow \ldots \Rightarrow \phi_{i_s}(p) \Rightarrow p \]
(By (4))

\[ \longrightarrow \text{Formula} \]

\[ f_1 \Rightarrow \ldots \Rightarrow f_r \Rightarrow \phi_{i_1}(p) \Rightarrow \ldots \Rightarrow \phi_{i_s}(p) \Rightarrow \top \]

is a tautology
(By (5))

\[ \longrightarrow \text{Formula} \]

\[ f_1 \Rightarrow \ldots \Rightarrow f_r \Rightarrow \phi_{i_1}(\bot) \Rightarrow \ldots \Rightarrow \phi_{i_s}(\bot) \Rightarrow \bot \]

is a tautology

\[ ^1 \text{In fact, only the } \longrightarrow \text{part is the deduction theorem. But, the } \iff \text{part is almost trivial.} \]
Formula
\[
f_1 \Rightarrow ... \Rightarrow f_r \Rightarrow \phi_{i1}(\bot) \Rightarrow ... \Rightarrow \phi_{is}(\bot) \Rightarrow \bot
\]
is a logical theorem
(By (5))

Formula
\[
f_1, ..., f_r, \phi_{i1}(\bot), ..., \phi_{is}(\bot) \vdash \bot
\]
holds.
(By (4))

\[
\mathcal{F}, \phi_1(\bot), \phi_2(\bot), ..., \vdash \bot
\]
which completes the proof. Proof of the \( \leftarrow \) part of (i) reads:
\[
\mathcal{F}, \phi_1(\bot), \phi_2(\bot), ..., \vdash \bot
\]

Formula
\[
f_1, ..., f_r, \phi_{i1}(\bot), ..., \phi_{is}(\bot) \vdash \bot
\]
we have: \( f_1, ..., f_r, \phi_{i1}(\bot), ..., \phi_{is}(\bot), ..., \vdash \bot \)
(Finiteness of every formal proof)

\[
\vdash f_1 \Rightarrow ... \Rightarrow f_r \Rightarrow \phi_{i1}(\bot) \Rightarrow ... \Rightarrow \phi_{is}(\bot) \Rightarrow \bot
\]
(By (4))

Formula
\[
f_1 \Rightarrow ... \Rightarrow f_r \Rightarrow \phi_{i1}(\bot) \Rightarrow ... \Rightarrow \phi_{is}(\bot) \Rightarrow \bot
\]
is a tautology
(By (5))

Formula
\[
f_1 \Rightarrow ... \Rightarrow f_r \Rightarrow \phi_{i1}(p) \Rightarrow ... \Rightarrow \phi_{is}(p) \Rightarrow p
\]
is a tautology

Formula
\[
f_1 \Rightarrow ... \Rightarrow f_r \Rightarrow \phi_{i1}(p) \Rightarrow ... \Rightarrow \phi_{is}(p) \Rightarrow p
\]
is a logical theorem
(By (5))
Formula
\[ f_1, \ldots, f_r, \phi_{i1}(p), \ldots, \phi_{is}(p) \vdash p \]
holds.
(By (4))

\[ \mathcal{F}, \phi_1(p), \phi_2(p), \ldots \vdash p \]
which completes the proof of (i).
We have omitted a proof of (ii) because (ii) can be proved in a similar way as (i).

Notice that Lemma 1 can be expressed by the following words:

*A literal \( \psi \) is a logical consequence of the given set if and only if the corresponding set is inconsistent.*

Now we prove the following lemma.

**Lemma 2.** The equivalence

\[ (7) \quad \mathcal{F}, p_1 \lor \ldots \lor p_k \vdash \bot \quad \implies \quad \mathcal{F} \vdash \neg p_1, \ldots, \mathcal{F} \vdash \neg p_k \]

(where \( p_i \) is any literal)

is true.

**Proof.** We have the following 'equivalence-chain':
\[ \mathcal{F}, p_1 \lor \ldots \lor p_k \vdash \bot \]

\[ \iff \mathcal{F} \vdash (p_1 \lor \ldots \lor p_k \implies \bot) \]

(By (4))

\[ \iff \mathcal{F} \vdash (\neg p_1 \land \ldots \land \neg p_k) \]

(Using a well-known tautology)

\[ \iff \mathcal{F} \vdash \neg p_1, \ldots, \mathcal{F} \vdash \neg p_k \]

which completes the proof.

Besides (6) and (7) we emphasize the following obvious equivalences

\[ (8) \quad \vdash \top \iff \mathcal{F}, \bot \vdash \bot \]

\[ (9) \quad \mathcal{F}, \top \vdash A \iff \mathcal{F} \vdash A \]

(\( A \) is a literal or the symbol \( \bot \))

\(^2\)A literal is an atom or the negation of an atom

\(^3\)i.e. one of the sets \( \mathcal{F}, \phi_1(\bot), \phi_2(\bot), \ldots \) or \( \mathcal{F}, \phi_1(\top), \phi_2(\top), \ldots \)
Suppose now that \( \mathcal{F} \) is a given set of clauses and \( \psi \) is a literal or \( \bot \). Is it possible that using the equivalences (6), (7), (8), (9) one can establish whether or not \( \psi \) is a logical consequence of \( \mathcal{F} \)? In order to answer this we introduce the following inference rules\(^4\)

(R1) \( \mathcal{F}, \bot \vdash \bot \iff \vdash \top \)

(R2) \( \mathcal{F}, \phi_1(p), \phi_2(p), \ldots \vdash p \iff \mathcal{F}, \phi_1(\bot), \phi_2(\bot), \ldots \vdash \bot \)
\( \mathcal{F}, \phi_1(p), \phi_2(p), \ldots \vdash \neg p \iff \mathcal{F}, \phi_1(\top), \phi_2(\top), \ldots \vdash \bot \)

(\( \phi_i(p) \) is any clause containing \( p \))

(R3) \( \mathcal{F}, p_1 \lor \ldots \lor p_k \vdash \bot \iff \mathcal{F} \vdash \neg p_1, \ldots, \mathcal{F} \vdash \neg p_k \)

(where \( p_i \) is any literal)

(R4) \( \mathcal{F}, \top \vdash A \iff \mathcal{F} \vdash A \)

(\( A \) is a literal or the symbol \( \bot \))

We emphasize that in the sequel for the set \( \mathcal{F} \) we suppose that it does not contain a clause of the form \( \ldots q \lor \neg q \ldots \), where \( q \) is any atom. Namely, such a formula is equivalent to \( \top \), consequently it should be omitted\(^5\). Similarly, if it happens that by applying rule (R2) some clause becomes equivalent to \( \top \) then we will also omit it.

Roughly speaking rules (R1),(R2),(R3),(R4) are used as follows:

_We start with a question (a sequent) of the form \( \mathcal{F} \vdash \psi \) and apply rules (R2),(R3),(R4) several times. If at some step we can apply rule (R1), the procedure stops with the conclusion that \( \psi \) is a logical consequence of \( \mathcal{F} \). However, if at some step we obtain the sequent \( \vdash \bot \) (then \( \mathcal{F} \) is an empty set) the procedure stops with the conclusion that \( \psi \) is not a logical consequence of \( \mathcal{F} \)._}

**Example 1.** Answer the following questions:

1) \( p \vdash p \)? 2) \( p, q \vdash p \)? 3) \( \vdash p \)? 4) \( q \vdash p \)?

5) \( \neg q \lor p, q \lor p \vdash p \)? 6) \( p, \neg p \lor q \lor \neg r, p \lor q \lor r \lor s, p \lor r \lor t \vdash \bot \)?

where \( p, q, r, s, t \) are atoms.

**Answer.**

1) Applying (R2) we obtain the sequent \( \bot \vdash \bot \) and by (R1) we get the sequent \( \vdash \top \) so the answer is: Yes.

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\(^4\)We point out that the set \( \mathcal{F} \) may be also an empty set.

\(^5\)This is compatible with rule (R4)
2) Applying (R2) we obtain a new question, i.e. the sequent \( \bot, q \vdash \bot \), and now applying (R1) we obtain the sequent \( \vdash \top \) so the answer is: Yes.
3) Applying (R2) we obtain the sequent \( \vdash \bot \) so the answer is: No.
4) By (R2) we obtain the sequent \( q \vdash \bot \) and after that by (R3) we obtain the sequent \( \vdash \lnot q \). Finally, by (R2) we obtain the sequent \( \vdash \bot \) such that the answer is: No.
5) By (R2) we obtain the sequent \( \lnot q, q \vdash \bot \). Now by (R3) applied to the literal \( \lnot q \) we obtain the sequent \( q \vdash q \), further by (R2) we obtain the sequent \( \bot \vdash \bot \) and finally by (R1) we obtain the sequent \( \vdash \top \) so the answer is: Yes.
6) Now by (R3) applied to clause \( p \) we obtain the sequent \( \lnot p \lor q \lor \lnot r, p \lor \lnot q \lor s, p \lor s \lor \lnot t \vdash \lnot p \).

By (R2) (and (R4) applied twice) we obtain the sequent \( q \lor \lnot r \vdash \bot \).

At this step applying (R3) we obtain two new sequents, i.e. questions \( \vdash \lnot q \) ? and \( \vdash r \) ?

The answer to the first question is No, so the final answer is also: No.

Concerning rules (R1)-(R4) we have this lemma.

**Lemma 3. (Soundness of rules (R1)-(R4)).** Let \( \mathcal{F} \) be any set of clauses. Suppose that we start with a sequent \( \mathcal{F} \vdash \psi \), where \( \psi \) is a literal or the symbol \( \bot \). If using rules (R1)-(R4) we obtain the sequent \( \vdash \top \) or the sequent \( \vdash \bot \), then \( \psi \) is / is not a logical consequence of set \( \mathcal{F} \), respectively.

**Proof** follows immediately from the fact that rules (R1)-(R4) are based on logical equivalences (6)-(9).

Let now \( \mathcal{F} \vdash \psi \) be any sequent. By \( Val(\mathcal{F} \vdash \psi) \) we denote its truth value, defined by:

- If \( \psi \) is a logical consequence of set \( \mathcal{F} \) then \( Val(\mathcal{F} \vdash \psi) \) is true
- otherwise \( Val(\mathcal{F} \vdash \psi) \) is false.

According to this definition and to rules (R1)-(R4), i.e. to equivalences (6)-(9) we have the following equalities

\[(10) \ Val(\vdash \top) = \text{true} \]
\[Val(\vdash \bot) = \text{false} \]
\[Val(\mathcal{F}, \bot \vdash \bot) = \text{true} \]
\[Val(\mathcal{F}, \top \vdash \psi) = Val(\mathcal{F} \vdash \psi) \]
\[Val(\mathcal{F}, \phi_1(p), \phi_2(p), \ldots \vdash p) = Val(\mathcal{F}, \phi_1(\bot), \phi_2(\bot), \ldots \vdash \bot) \]
\[ Val(\mathcal{F}, \phi_1(p), \phi_2(p), \ldots, \vdash \neg p) = Val(\mathcal{F}, \phi_1(T), \phi_2(T), \ldots, \vdash \bot) \]

(\(\phi_i(p)\) is any clause containing \(p\))

\[ Val(\mathcal{F}, p_1 \lor \ldots \lor p_k, \vdash \bot) \]

\[ = Val(\mathcal{F}, \vdash \neg p_1) \text{ and } \ldots \text{ and } Val(\mathcal{F}, \vdash \neg p_k) \]

(where \(p_i\) is any literal, i.e. an atom or the negation of an atom)

Suppose that \(\mathcal{F}\) is a finite set. Then, in fact, these equalities define the function \(Val\) recursively on the number of all member of set \(\mathcal{F}\). Consequently, these equalities suggest how to calculate \(Val(\mathcal{F}, \vdash \psi)\). In other words we have the following assertion:

(11) If \(\mathcal{F}\) is a finite set then one can effectively calculate \(Val(\vdash \psi)\), i.e. establish whether or not \(\psi\) is a logical consequence of set \(\mathcal{F}\).

Next we will prove the following basic theorem.

**Theorem 1. (Completeness)** Let \(\mathcal{F}\) be a set of some clauses and \(\psi\) a literal or the symbol \(\bot\). Then:

\(\psi\) is a logical consequence of set \(\mathcal{F}\) if and only if starting with \(\mathcal{F}, \vdash \psi\) and applying rules (R1)-(R4) a finite number of times one can obtain the sequent \(\vdash T\).

**Proof.** The if - part follows immediately from Lemma 3. To prove the only if - part suppose now that \(\psi\) is a logical consequence of set \(\mathcal{F}\). Then \(\psi\) is a logical consequence of some finite subset \(A\) of set \(\mathcal{F}\) (for: every formal proof is finite). Next, by (11) we conclude that starting with the sequent \(A, \vdash \psi\) and applying rules (R1)-(R4) a finite number of times one can obtain the sequent \(\vdash T\). Consequently, also starting with the sequent \(\mathcal{F}, \vdash \psi\) and applying rules (R1)-(R4) a finite number of times one can obtain the sequent \(\vdash T\). The proof is complete.

**REFERENCES**

