EQUATIONAL REFORMULATION OF FORMAL THEORIES

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1. There are many important instances of formal theories (cf., for example, [4]), as propositional calculi, predicate calculi, formal arithmetics, axiomatic set theories and so on. Within the theory of universal algebras the concept of variety is of particular interest (cf. for example [2]). Every variety, with appropriate precision introduced, becomes a formal theory. Formal theories of this kind contain formulae of the form \( t_1 = t_2 \), where \( t_1 \) and \( t_2 \) are terms (constructed out of some primitive symbols, constants, individual variables and operation symbols; cf. [2]). Axioms are some formulae given in advance, as formulae of the form \( t = t \), where \( t \) is any term. The rules of inference are (in agreement with elementary properties of equality):

\[
\frac{t_1 = t_2}{t_2 = t_1} \quad \frac{t_1 = t_2, t_2 = t_3}{t_1 = t_3} \quad \frac{t_1 = t_1', \ldots, t_n = t_n'}{\omega t_1 \ldots t_n = \omega t_1' \ldots t_n'}
\]

(where \( t_i \) is any term and \( \omega \) an operation symbol of length \( n \)).

A formal theory of this kind we shall call \textit{equational formal theory}. One of our aims is to investigate the connection between equational and other formal theories.

2. Let \( \mathcal{T} \) be a formal theory with axioms \( A_i \) (\( i \in I; \ I \) is a given set of indexes). By \( \mathcal{T} (\sim) \) we denote the equational theory defined as follows:

1° The formulae of \( \mathcal{T} \) play the role of individual variables of \( \mathcal{T} (\sim) \); the symbol \( \& \) is an operation symbol\(^{1)} \) of length 2.

2° The axioms of \( \mathcal{T} (\sim) \) are formulae of the form

\[
\begin{align*}
(2) & \quad A_i \sim \top \quad (\top \text{ is an arbitrarily chosen axiom of } \mathcal{T}; \ i \in I), \\
(b) & \quad A \sim A, \ & AB \sim & BA, \ & \& ABC \sim & A & BC \text{ and } \ A \top \sim A \\
& \quad (A, B \text{ and } C \text{ are terms of } \mathcal{T} (\sim)).
\end{align*}
\]

\(^{1)}\) The terms of \( \mathcal{T} (\sim) \) satisfy the following definition: (i) formulae of \( \mathcal{T} \) are terms of \( \mathcal{T} (\sim) \); (ii) if \( A \) and \( B \) are terms, then \( \& AB \) is a term; (iii) every term is obtained by a finite number of applications of (i) and (ii).

The formulae of \( \mathcal{T} (\sim) \) are of the form \( A \sim B \), where \( A \) and \( B \) are terms and \( \sim \) and \( \& \) are not among the symbols of \( \mathcal{T} \).
(c) Let \[
\Phi_1, \ldots, \Phi_k \quad \frac{\Phi}{\Phi}
\]
be any rule on inference of \(\mathcal{T}\); then the formula
\[
\& \ldots \& \Phi_1 \ldots \Phi_k \sim \& \ldots \& \Phi_1 \ldots \Phi_k \Phi
\]
is an axiom of \(\mathcal{T}(\sim)\).

3° The rule of inference of \(\mathcal{T}(\sim)\) are
\[
\begin{align*}
\frac{A \sim B}{B \sim A}, & \quad \frac{A \sim B, B \sim C}{\sim A}, & \quad \frac{A_1 \sim B_1, A_2 \sim B_2}{\sim A_1 A_2 \sim B_1 B_2}
\end{align*}
\]
\((A, B, \ldots \) are terms of \(\mathcal{T}(\sim))\).

Note. In the sequel we shall write \(A \& B, A \& B \& C\) etc., instead of \(A \& B, A \& B, A \& B \& C\) etc., respectively. The axiom (b) prevents us from possible confusion. For example, in this case axiom (c) becomes: \(\Phi_1 \& \ldots \& \Phi_k \sim \Phi_1 \& \ldots \& \Phi_k \& \Phi\).

As we shall see soon, the symbol \(\&\) is related to the metatheoretic and while the symbol \(\sim\) is, so to say, a formalization of the relation that we call equiconsequence. In fact, we prove

**Theorem 1.** Let \(P_1, \ldots, P_r, Q_1, \ldots, Q_s\) be formulae of \(\mathcal{T}\); then
\[
\vdash P_1 \& \ldots \& P_r \sim Q_1 \& \ldots \& Q_s \quad \text{iff} \quad P_1, \ldots, P_r \vdash Q_1, \ldots, Q_s \quad \text{and}
\]
\[
Q_1, \ldots, Q_s \vdash P_1, \ldots, P_r.
\]

We shall prove two lemmata first.

**Lemma 1.** If \(P_1, \ldots, P_r \vdash Q\), then \(\vdash P_1 \& \ldots \& P_r \sim P_1 \& \ldots \& P_r \& Q\)

where \(P_i\) and \(Q\) are formulae of \(\mathcal{T}\).

**Proof.** We use induction on the length \(n\) of the shortest proof of \(P_1, \ldots, P_r \vdash Q\).

**Case** \(n = 1\). \(Q\) is either \(P_i\) (for some \(1 \leq i \leq r\)) or \(A_j\) (for some \(j \in I\)). In both cases we have
\[
\begin{align*}
\vdash P_1 \& \ldots \& P_r \sim P_1 \& \ldots \& P_r \& Q
\end{align*}
\]
(for we have
\[
\begin{align*}
\vdash P_1 \& \ldots \& P_r \sim P_1 \& \ldots \& P_r \& P_i
\end{align*}
\]
\[
\vdash P_1 \& \ldots \& P_r \sim P_1 \& \ldots \& P_r \& \top, \quad \vdash A_j \sim \top.
\]

**Case** \(n > 1\). The following subcases are possible:

(i) \(Q\) is \(P_i\) (for some \(1 \leq i \leq r\));

(ii) \(Q\) is \(A_j\) (for some \(j \in I\));
(iii) \( Q \) is a consequence of some preceding formulae by a rule

\[
\frac{\Phi_1, \ldots, \Phi_k}{\Phi}
\]

In both (i) and (ii) we proceed as in Case \( n = 1 \). In (iii) by induction hypothesis we have

\[
\vdash_{\text{\( n = 1 \)}} P_1 \land \ldots \land P_r \sim P_1 \land \ldots \land P_r \land \Phi_1
\]

\[
\vdots
\]

\[
\vdash_{\text{\( n = 1 \)}} P_1 \land \ldots \land P_r \sim P_1 \land \ldots \land P_r \land \Phi_k.
\]

Therefrom we derive immediately

\[
(3) \quad \vdash_{\text{\( n = 1 \)}} P_1 \land \ldots \land P_r \sim P_1 \land \ldots \land P_r \land \Phi_1 \land \ldots \land \Phi_k
\]

i.e.,

\[
(4) \quad \vdash_{\text{\( n = 1 \)}} P_1 \land \ldots \land P_r \sim P_1 \land \ldots \land P_r \land \Phi_1 \land \ldots \land \Phi_k \land \Phi
\]

[for

\[
\vdash_{\text{\( n = 1 \)}} \Phi_1 \land \ldots \land \Phi_k \sim \Phi_1 \land \ldots \land \Phi_k \land \Phi
\]

From (3) and (4) we derive

\[
\vdash_{\text{\( n = 1 \)}} P_1 \land \ldots \land P_r \sim P_1 \land \ldots \land P_r \land \Phi
\]

i.e.

\[
\vdash_{\text{\( n = 1 \)}} P_1 \land \ldots \land P_r \sim P_1 \land \ldots \land P_r \land Q \quad \text{(for } Q \text{ is } \Phi)\]

Lemma 2. If \( P_1, \ldots, P_r \vdash Q_1, \ldots, Q_s \), then

\[
\vdash_{\text{\( n = 1 \)}} P_1 \land \ldots \land P_r \sim P_1 \land \ldots \land P_r \land Q_1 \land \ldots \land Q_s.
\]

This lemma is an immediate consequence of Lemma 1. Indeed, from \( P_1, \ldots, P_r \vdash Q_1, \ldots, Q_s \), by Lemma 1, it follows that

\[
\vdash_{\text{\( n = 1 \)}} P_1 \land \ldots \land P_r \sim P_1 \land \ldots \land P_r \land Q_1
\]

\[
\vdots
\]

\[
\vdash_{\text{\( n = 1 \)}} P_1 \land \ldots \land P_r \sim P_1 \land \ldots \land P_r \land Q_s
\]

are theorems; hence, \( \vdash_{\text{\( n = 1 \)}} P_1 \land \ldots \land P_r \sim P_1 \land \ldots \land P_r \land Q_1 \land \ldots \land Q_s \).
Now, we shall prove Theorem 1.

The "if" part. Suppose that \( P_1, \ldots, P_r \vdash Q_1, \ldots, Q_s \) and \( \vdash P_1, \ldots, P_r \). Therefrom, by Lemma 2,
\[
\vdash P_1 \& \ldots \& P_r \& \sim P_1 \& \ldots \& P_r \& \sim Q_1 \& \ldots \& Q_s
\]
and
\[
\vdash Q_1 \& \ldots \& Q_s \& \sim Q_1 \& \ldots \& Q_s \& \sim P_1 \& \ldots \& P_r
\]
and hence
\[
\vdash P_1 \& \ldots \& P_r \& \sim Q_1 \& \ldots \& Q_s
\]

The "only if" part. Suppose that \( \vdash P_1 \& \ldots \& P_r \& \sim Q_1 \& \ldots \& Q_s \) and let \( A_1, \ldots, A_p, B_1, \ldots, B_q \) be arbitrary formulae of \( \mathcal{T} \). Let us associate the sequents
\[
A_1, \ldots, A_p \vdash B_1, \ldots, B_q \text{ and } B_1, \ldots, B_q \vdash A_1, \ldots, A_p
\]
to the formula
\[
A_1 \& \ldots \& A_p \& \sim B_1 \& \ldots \& B_q
\]
and let \( \Psi \) denote this association.

Applying the mapping \( \Psi \) to the axioms of \( \mathcal{T} (\sim) \) we obtain proofs from hypotheses in \( \mathcal{T} \). For example, such proofs from hypotheses are \( \vdash \frac{T \vdash A_i; \ A, T \vdash A; \ \Phi_i, \ldots, \Phi_k \vdash \Phi_i, \ldots, \Phi_k, \ \Phi \text{ and so on.}}{\mathcal{T}} \)

Moreover, the mapping \( \Psi \) is in accordance with rules of \( \mathcal{T} (\sim) \) — in fact, the rules of \( \mathcal{T} (\sim) \) are translated into true statements about proofs from hypotheses in \( \mathcal{T} \). For example, to the rule
\[
\frac{A \& \sim B, B \& \sim C}{A \& \sim C}
\]
there corresponds the statement

If \( \vdash A, \ B \vdash A, \ B \vdash C, \ C \vdash B \), then \( \vdash C, \ C \vdash A \).

In accordance with consideration, if we apply \( \Psi \) to the supposed theorem
\[
P_1 \& \ldots \& P_r \& \sim Q_1 \& \ldots \& Q_s
\]
we obtain proofs from hypotheses
\[
P_1, \ldots, P_r \vdash Q_1, \ldots, Q_s \text{ and } Q_1, \ldots, Q_s \vdash P_1, \ldots, P_r
\]

This completes the proof of the theorem.

According to Theorem 1, just proved, we can say that in a sense \( \mathcal{T} (\sim) \) is a formalization of deduction relation of \( \mathcal{T} \). In particular, by Theorem 1 it follows that
\[
\vdash A, \ B \vdash A \iff \vdash A \& \sim B.
\]
3. By the next theorem a connection is established between the theorems of \( \mathcal{T} \) and some theorems of \( \mathcal{T}(\sim) \).

**Lemma 3.** Let \( A \) be any formula of \( \mathcal{T} \); then

\[
\vdash A \ \text{iff} \quad \vdash_A A \sim \top \quad (\mathcal{T})
\]

\[
\vdash \mathcal{T}(\sim)
\]

**Proof.** \( \vdash A \) iff \( A \vdash \top \), \( \top \vdash A \) (by definition of \( \vdash \))

\[
\text{iff} \quad \vdash A \sim \top \quad (\text{by Theorem 1})
\]

Hence, \( \vdash A \) iff \( \vdash A \sim \top \).

Let \( f \) denote a mapping of the set For (\( \mathcal{T} \)) (the set of formulae of \( \mathcal{T} \)) into the set For (\( \mathcal{T}(\sim) \)), defined by equality

\[
f(A) \overset{\text{def}}{=} A \sim \top.
\]

According to Lemma 3, by the injective mapping \( f \) the set of theorems of \( \mathcal{T} \) is mapped into the set of theorems of \( \mathcal{T}(\sim) \). Moreover, the mapping translates the proofs of \( \mathcal{T} \) into (incomplete, but completable) proofs of \( \mathcal{T}(\sim) \). In fact:

(i) if \( A_i \) is an axiom of \( \mathcal{T} \), then \( f(A_i) \), i.e. \( A_i \sim \top \) is a theorem of \( \mathcal{T} \);

(ii) if

\[
\Phi_1, \ldots, \Phi_k
\]

\( \Phi \)

is a rule of \( \mathcal{T} \), then in \( \mathcal{T}(\sim) \) it is the case that\(^2\)

\[
f(\Phi_1), \ldots, f(\Phi_k) \vdash f(\Phi)
\]

i.e.

\[
\Phi_1 \sim \top, \ldots, \Phi_k \sim \top \vdash \Phi \sim \top
\]

Having in mind the properties of the map \( f \) (it is \( 1-1 \), it translates theorems and proofs of \( \mathcal{T} \) into theorems and proofs of \( \mathcal{T}(\sim) \)) we can say:

\( \mathcal{T} \) is isomorphically embedded in \( \mathcal{T}(\sim) \) by the mapping \( f \).

In this way we conclude that the following theorem is valid.

**Theorem 2.** Any formal theory can be isomorphically embedded in an equational formal theory.

4. Let \( \mathcal{T} \) be a formulation of the classical propositional calculus, say \( P_2 \) of [1]. The axioms (inessentially modified) are formulae of the form\(^3\)

\[
A \Rightarrow (B \Rightarrow A), \ (A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)), \ (\top \Rightarrow A \Rightarrow \top) \Rightarrow (B \Rightarrow A)
\]

\( A, B, C \) are propositional formulas, \( A \Rightarrow B \) and \( \top \) are defined in terms of the primitive ones: for example, \( A \lor B \) stand for \( (A \Rightarrow B) \Rightarrow B \), etc.

\(^2\) Indeed, let \( \Phi_1 \sim \top, \ldots, \Phi_k \sim \top \) be hypotheses. Using then we obtain \( \Phi_1 \& \ldots \& \Phi_k \sim \top \) and \( \top \& \ldots \& \top \), i.e. \( \Phi_1 \& \ldots \& \Phi_k \sim \top \). But we have \( \Phi_1 \& \ldots \& \Phi_k \sim \Phi_1 \& \ldots \& \Phi_k \& \Phi \) and hence \( \Phi_1 \& \ldots \& \Phi_k \& \Phi \sim \top \). Therefore, \( \top \& \Phi \sim \top \) and finally, \( \Phi \sim \top \).

\(^3\) The primitive connectives are \( \Rightarrow \) and \( \top \). The connectives \( \land, \lor \) and \( \Rightarrow \) are defined in terms of the primitive ones: for example, \( A \lor B \) stand for \( (A \Rightarrow B) \Rightarrow B \), etc.
The only rule is modus ponens:

\[
\frac{A, A \Rightarrow B}{B}
\]

We prove

**Theorem 3.** Let \( A_1, \ldots, A_p, B_1, \ldots, B_q \) be any formulas of \( P_2 \), then

\[
\vdash_{P_2(\sim)} A_1 \& \cdots \& A_p \sim B_1 \& \cdots \& B_q \iff \vdash_{P_2} A_1 \land \cdots \land A_p \Rightarrow B_1 \land \cdots \land B_q.
\]

**Proof.** \( \vdash_{P_2(\sim)} A_1 \& \cdots \& A_p \sim B_1 \& \cdots \& B_q \iff \)

\[
A_1, \ldots, A_p \vdash_{P_2} B_1, \ldots, B_q \quad \text{and} \quad B_1, \ldots, B_q \vdash_{P_2} A_1, \ldots, A_p
\]

(by Theorem 1); but this is the case iff

\[
A_1 \land \cdots \land A_p \vdash_{P_2} B_1 \land \cdots \land B_q \quad \text{and} \quad B_1 \land \cdots \land B_q \vdash_{P_2} A_1 \land \cdots \land A_p
\]

(this is provable in \( P_2 \)); again, this is the case iff

\[
\vdash_{P_2} A_1 \land \cdots \land A_p \Rightarrow B_1 \land \cdots \land B_q \quad \text{and} \quad \vdash_{P_2} B_1 \land \cdots \land B_q \Rightarrow A_1 \land \cdots \land A_p
\]

(by deduction theorem); by definition of \( \Rightarrow \), this is the case iff

\[
\vdash_{P_2} A_1 \land \cdots \land A_p \Leftrightarrow B_1 \land \cdots \land B_q.
\]

Let us note that the preceding proof relies on the fact that in \( \mathcal{T} \) viz. \( P_2 \) the following conditions are satisfied:

**Condition 1.** There is an operation in \( \mathcal{T} \), in symbols \( \land \), such that \( \vdash_{\mathcal{T}} A, B \vdash_{\mathcal{T}} A \land B \) and \( A \land B \vdash_{\mathcal{T}} A, B \) (\( A, B \) are formulas of \( \mathcal{T} \)).

**Condition 2.** There is an operation in \( \mathcal{T} \), in symbols \( \Rightarrow \), such that \( \vdash_{\mathcal{T}} A \Rightarrow B \) iff \( \vdash_{\mathcal{T}} A \Rightarrow B \) (\( A, B \) are formulas of \( \mathcal{T} \)).

According to Theorem 3, to any theorem

\[
A_1 \& \cdots \& A_p \sim B_1 \& \cdots \& B_q
\]

of \( P_2(\sim) \) there corresponds the theorem

\[
A_1 \land \cdots \land A_p \Leftrightarrow B_1 \land \cdots \land B_q
\]

of \( P_2 \). In other words, by substituting \( \land \) and \( \Leftrightarrow \) for \( \& \) and \( \sim \), respectively, the formulas of \( P_2(\sim) \) are translated into formulas of \( P_2 \), and, moreover, theorems are translated into theorems. Also, (this is proved easily), by this injective mapping the proofs of \( P_2(\sim) \) are translated into (completable) proofs
of \( P_2 \). On the other hand, the converse is also true in a sense; for example, to any theorem \( A \) of \( P_2 \) there corresponds (by Lemma 3) the theorem \( A \sim \top \) of \( P_2(\sim) \). Therefore, \( P_2(\sim) \) is isomorphically embedded in \( P_2 \). The calculus \( P_2(\sim) \) we shall also call an equational reformulation of \( P_2 \).

Remark. Let us note that the axioms of \( P_3(\sim) \) can be transformed into axioms of Boolean algebra (cf. for example, [3], p. 5)

\[
A \land \top \sim A, \quad A \lor \top \sim A
\]

\[
A \land \top \sim A, \quad A \lor \top \sim \top
\]

\[
A \land B \sim B \land A, \quad A \lor B \sim B \lor A
\]

\[
A \land (B \lor C) \sim (A \land B) \lor (A \land C), \quad A \lor (B \land C) \sim (A \lor B) \land (A \lor C)
\]

and, in addition\(^4\)

\[
(A \land B \sim A \lor B)
\]

\((A, B, C \text{ are any formulas of } P_2; \top \text{ is, say, } p \Rightarrow (p \Rightarrow p)).\)

Proof. Using axioms and rules of \( P_2(\sim) \), we prove easily (2), (5), (6).

The formula (5) can be proved as follows. We have

\[
A, B \vdash A \land B, \quad A \land B \vdash A, B
\]

and, hence, by Theorem 1

\[
\vdash A \lor B \sim A \land B.
\]

Furthermore, the proof of, say

\[
\begin{align*}
A \sim B \\
\top A \sim \top B
\end{align*}
\]

is as follows. Suppose that \( \vdash A \sim B \); then according to Theorem 3,

\[
\vdash A \Leftrightarrow B.
\]

Hence, using the well-known properties of \( P_2 \), we conclude that

\[
\vdash \top A \Leftrightarrow \top B,
\]

and hence, again by Theorem 3, we obtain \( \vdash \top A \Leftrightarrow \top B \).

Let us assume now that (2), (5), and (6) hold and let us prove the axioms and rules of \( P_2(\sim) \). Using (2), (5), and (6) we can prove various facts about Boolean algebra, such as

\[
\top \top A \sim A, \quad \top (A \land B) \sim A \lor B, \quad A \Rightarrow B \sim \top A \lor B \text{ etc.}
\]

\(^4\) Besides the axioms given above, we assume a number of properties of equality (\( \sim \) stands for \( = \)):

\[
\begin{align*}
A \sim A, & \quad A \sim B, B \sim C \quad \frac{A \sim B}{B \sim A}, \quad A \sim C \\
A \sim B, & \quad A \sim B, B \sim C \quad \frac{A \sim B}{B \sim A}, \quad A \sim C \\
A \sim B, C \sim D & \quad \frac{A \sim B}{B \sim A}, \quad A \sim C, B \sim D \quad \frac{A \sim B, C \sim D}{A \sim B, C \sim D}
\end{align*}
\]
Using the last formula, we easily prove formulas

\[ A \supset (B \supset A) \sim \top, \quad (A \supset (B \supset C) \supset ((A \supset B) \supset (A \supset C)) \sim \top, \]

\[ (\top \supset \top \supset B) \supset (B \supset A) \sim \top \]

i.e. a number of axioms of \( P_2 (\sim) \). These axioms are of the form (2) (a). The axioms of the form (2) (b) are proved easily, using (5). In a similar way we prove axioms of the form (3) (c), i.e. the formula \( A \& (A \supset B) \sim A \& (A \supset B) \& B \).

Let \( \mathcal{T} \) be a formal theory satisfying conditions 1. and 2. This means that the symbols \( \&, \supset \) are either primitive in \( \mathcal{T} \) or defined\(^5\) such that we have 1. and 2. viz.

\[ A, B \vdash A \& B \quad \text{and} \quad A \& B \vdash A, B \]

\[ A \vdash B \quad \text{iff} \quad \vdash A \supset B \]

Then we have the following theorem which is proved almost in the same way as in the case of \( P_2 \).

**Theorem 4.**

1° \[ \vdash A \quad \text{iff} \quad \vdash A \sim \top \]

2° \[ \vdash A \iff B \quad \text{iff} \quad \vdash A \sim B. \]

In other words, if the conditions 1. and 2. are satisfied, \( \mathcal{T}(\sim) \) is an equational reformulation of \( \mathcal{T} \).

Finally, let us note that there are various formal theories satisfying conditions 1. and 2. — for example, the classical propositional calculus, the intuitionistic propositional calculus and many others.

**References**


\[ ^5 \] Then, for example, \( A \& B \) stand for a formula constructed in some way out of subformulas of \( A \) and \( B \).