AN ALGORITHM FOR THE SOLUTION OF A 2 × 2 SYSTEM OF NONLINEAR ALGEBRAIC EQUATIONS

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Summary

This paper treats the problem of simultaneous determination of all solutions of the system of algebraic equations

\[ J_1(x, y) = A_1x^2 + 2B_1xy + C_1y^2 + 2D_1x + 2E_1y + F_1 = 0, \]
\[ J_2(x, y) = A_2x^2 + 2B_2xy + C_2y^2 + 2D_2x + 2E_2y + F_2 = 0, \]

on the assumption that they are different. The algorithm is based on the use of basic ideas of an iterative procedure for factorisation of polynomials given in papers [1] and [2]. This analysis would be preliminarily made and its analogy of treatment would be transferred to the construction of algorithm for the solution of given problem. For treatment of this kind of problem immediate cause was conditioned by practical needs.

Introduction and statement of the problem

In papers [1] and [2] an iterative procedure for simultaneous determination of all roots of an algebraic polynomial was given, under the assumption that they differ one from another.

For example, in the case of an algebraic equation

\[ P(x) = x^3 + p_2x^2 + p_1x + p_0 = 0, \]

whose roots are \(a, b, c\), it was demonstrated that they represent limit values of a series \(a_n, b_n, c_n\) defined as follows

\[ a_{n+1} = a_n - \frac{P(a_n)}{(a_n - b_n)(a_n - c_n)}, \]
\[ b_{n+1} = b_n - \frac{P(b_n)}{(b_n - a_n)(b_n - c_n)}, \]
\[ c_{n+1} = c_n - \frac{P(c_n)}{(c_n - a_n)(c_n - b_n)}. \]
For starting data \( a_0, b_0, c_0 \), approximative values of roots \( a, b, c \) are selected.

Otherwise, formulae (2) are derived from following polynomial identity

\[
(x - a_{n+1})(x - b_n)(x - c_n) + (x - a_n)(x - b_{n+1})(x - c_n) + 
+x - a_n)(x - b_n)(x - c_{n+1}) - 2(x - a_n)(x - b_n)(x - c_n) = P(x),
\]

that may be written in this way

\[
(x - a_n)(x - b_n)(x - c_n) - (a_{n+1} - a_n)(x - b_n)(x - c_n) - 
(b_{n+1} - b_n)(x - c_n)(x - a_n) - (c_{n+1} - c_n)(x - a_n)(x - b_n) = P(x).
\]

Now, let's observe the set of all polynomial expressions of \( a_n, b_n, c_n \). Various constants, as well as a variable \( x \), may take part here. For example such is the case for

\[
a_n + b_n, \ a_n b_n + 3 c_n, \ x - a_n, \ (x - a_n)(x - b_n)(x - c_n), \ etc.
\]

In this set we define the operator \( \delta \) in following way

\[
\begin{align*}
\delta a_n & \overset{\text{def}}{=} a_{n+1} - a_n, \quad \delta b_n \overset{\text{def}}{=} b_{n+1} - b_n, \quad \delta c_n \overset{\text{def}}{=} c_{n+1} - c_n, \\
\delta u & \overset{\text{def}}{=} 0, \quad (u \text{ is an expression which does not include index } n) \\
\delta (u + v) & = \delta u + \delta v, \quad \delta (uv) = v \delta u + u \delta v \quad (u \text{ and } v \text{ are the expressions whatsoever}); \text{ it means that for the operator } \delta \text{ similar formulae are valuable as for the differentiation. For example:}
\end{align*}
\]

\[
\begin{align*}
\delta (a_n + b_n) & = \delta a_n + \delta b_n = a_{n+1} - a_n + b_{n+1} - b_n, \\
\delta (x - a_n) & = \delta x - \delta a_n = 0 - (a_{n+1} - a_n) = a_n - a_{n+1}, \\
\delta [(x - a_n)(x - b_n)(x - c_n)] & = (x - b_n)(x - c_n) \delta (x - a_n) + \\
&+ (x - a_n)(x - c_n) \delta (x - b_n) + (x - a_n)(x - b_n) \delta (x - c_n) = \\
&= (x - b_n)(x - c_n)(a_n - a_{n+1}) + (x - a_n)(x - c_n)(b_n - b_{n+1}) + \\
&+ (x - a_n)(x - b_n)(c_n - c_{n+1}).
\end{align*}
\]

On the basis of last equality (from above mentionned examples) the formula (4) may be written in this way

\[
(x - a_n)(x - b_n)(x - c_n) + \delta [(x - a_n)(x - b_n)(x - c_n)] - P(x) = 0.
\]

Introducing the designation

\[
Q_n = (x - a_n)(x - b_n)(x - c_n) - P(x),
\]

the equality (6) may be presented in the from

\[
Q_n + \delta Q_n = 0,
\]

as we have

\[
Q_n + \delta Q_n = (x - a_n)(x - b_n)(x - c_n) - P(x) + \delta [(x - a_n)(x - b_n)(x - c_n) - \\
- P(x)] = (x - a_n)(x - b_n)(x - c_n) - P(x) + \\
+ \delta [(x - a_n)(x - b_n)(x - c_n)] - \delta P(x) = (x - a_n)(x - b_n)(x - c_n) + \\
+ \delta [(x - a_n)(x - b_n)(x - c_n)] - P(x), \quad (\delta P(x) = 0).
\]
Basic method

Let's now consider the system of equations

\begin{align*}
J_1(x, y) &= A_1x^2 + 2B_1xy + C_1y^2 + 2D_1x + 2E_1y + F_1 = 0, \\
J_2(x, y) &= A_2x^2 + 2B_2xy + C_2y^2 + 2D_2x + 2E_2y + F_2 = 0,
\end{align*}

supposing that its solutions are \((a, \alpha), (b, \beta), (c, \gamma), (d, \delta)\).

Following schema for further treatment may be useful

\[ \] \begin{center}
\includegraphics[width=0.5\textwidth]{schema.png}
\end{center}

The system of equations (8) is equivalent to the system of equations

\begin{align*}
AB \cdot CD &= 0, \\
AC \cdot BD &= 0,
\end{align*}

where \(AB, \ldots, BD\) denote following expressions

\[
AB = (\beta - \alpha)(x - a) - (b - a)(y - \alpha), \quad \text{def}
\]

\[
BD = (\delta - \beta)(x - b) - (d - b)(y - \beta), \quad \text{def}
\]

Naturally, \(AB = 0, \ldots, BD = 0\) are the equations of straight lines through the points \(A\) and \(B, \ldots,\) or \(B\) and \(D\) respectively.

Equations of the system (9) are certain linear combinations equation of system (8); in other words there exists certain constantes \(\lambda, \mu, \rho, \varphi\) so that following equalities are valable

\begin{align*}
AB \cdot CD + \lambda J_1(x, y) + \mu J_2(x, y) &= 0, \\
AC \cdot BD + \rho J_1(x, y) + \varphi J_2(x, y) &= 0.
\end{align*}

(10)

Basic idea of algorithm given in this paper is this: iterative procedure of the system of equations (8) in the limit shall result in the form (10). Pursuant this point, preliminarily we introduce the designations

\[
R_n = A_nB_n \cdot C_nD_n + \lambda_n J_1(x, y) + \mu_n J_2(x, y),
\]

\[
S_n = A_nC_n \cdot B_nD_n + \rho_n J_1(x, y) + \varphi_n J_2(x, y).
\]
where $A_nB_n, \ldots, B_nD_n$ are following expressions

$$A_nB_n = (\beta_n - a_n) (x - a_n) - (b_n - a_n) (y - \alpha_n),$$
$$C_nD_n = (\delta_n - \gamma_n) (x - c_n) - (d_n - c_n) (y - \gamma_n),$$
$$A_nC_n = (\gamma_n - \alpha_n) (x - a_n) - (c_n - a_n) (y - \alpha_n),$$
$$B_nD_n = (\delta_n - \beta_n) (x - b_n) - (d_n - b_n) (y - \beta_n).$$

In expression for $R_n$ and $S_n$, 12 series are participating

$$a_n, \alpha_n, b_n, \beta_n, c_n, \gamma_n, d_n, \delta_n, \lambda_n, \nu_n, \rho_n, \varphi_n.$$

Our aim is to define these series so that

$$a_n \rightarrow a, \alpha_n \rightarrow \alpha, \ldots, \varphi_n \rightarrow \varphi,$$

when $n \rightarrow \infty$.

Conformably to equation (7) which refer to algebraic polynomials, for $R_n$ and $S_n$ following conditions are posed

$$R_n + \delta R_n = 0,$$
$$S_n + \delta S_n = 0.$$  \hspace{1cm} (11)

Now, operator $\delta$ refers to polynomial expressions for

$$a_n, \alpha_n, b_n, \beta_n, \ldots, \varphi_n, \varphi_n$$

and is introduced with equalities similar to equalities (5).

On the basis of the definition of operator $\delta$ we have

$$\delta R_n = \delta [A_nB_n \cdot C_nD_n + \lambda_n J_1 (x, y) + \mu_n J_2 (x, y)] = C_nD_n \delta A_nB_n +$$
$$+ A_nB_n \delta C_nD_n + J_1 (x, y) \delta \lambda_n + J_2 (x, y) \delta \mu_n =$$

$$= [(\delta_n - \gamma_n) (x - c_n) - (d_n - c_n) (y - \gamma_n)] \cdot \delta [(\beta_n - \alpha_n) (x - a_n) -$$
$$- (b_n - a_n) (y - \alpha_n)] + [(\beta_n - \alpha_n) (x - a_n) -$$
$$- (b_n - a_n) (y - \alpha_n)] \cdot \delta [(\delta_n - \gamma_n) (x - c_n) - (d_n - c_n) (y - \gamma_n)] +$$
$$+ J_1 (x, y) \delta \lambda_n + J_2 (x, y) \delta \mu_n = [(\beta_n+1 - \beta_n - \alpha_{n+1} + \alpha_n) (x - a_n) -$$
$$- (\beta_n - \alpha_n) (a_{n+1} - a_n) - (b_{n+1} - b_n) - (a_{n+1} + a_n) (y - \alpha_n) +$$
$$+ (b_n - a_n) (\alpha_{n+1} - \alpha_n)] \cdot [(\delta_n - \gamma_n) (x - c_n) - (d_n - c_n) (y - \gamma_n)] +$$
$$+ [(\delta_n+1 - \delta_n - \gamma_{n+1} \gamma_n) (x - c_n) - (\delta_n - \gamma_n) (c_{n+1} - c_n) -$$
$$- (d_{n+1} - d - c_{n+1} + c_n) (y - \gamma_n) + (d_n - c_n) (\gamma_{n+1} - \gamma_n)] \cdot [(\beta_n - \alpha_n) (x - a_n) -$$
$$- (b_n - a_n) (y - \alpha_n)] + (\lambda_{n+1} - \lambda_n) J_1 (x, y) + (\mu_{n+1} - \mu_n) J_2 (x, y).$$

Similar procedure lead us to the expression for $\delta S_n$, too. Using derived expressions for $\delta R_n$ and $\delta S_n$ equalities (11) become
An algorithm for the solution of a system of nonlinear algebraic equations

\[
\begin{align*}
[(\beta_n - \alpha_n) (x - a_n) - (b_n - a_n) (y - \alpha_n)] \cdot & [(\delta_n - \gamma_n) (x - c_n) - (d_n - c_n) (y - \gamma_n)] + \\
+ & [(\beta_{n+1} - \beta_n - \alpha_{n+1} + \alpha_n) (x - a_n) - (\beta_n - \alpha_n) (a_{n+1} - a_n) - \\
- & (b_{n+1} - b_n - a_{n+1} + a_n) (y - \alpha_n) + (b_n - a_n) (\alpha_{n+1} - \alpha_n)] \cdot [(\delta_n - \gamma_n) (x - c_n) - \\
- & (d_n - c_n) (y - \gamma_n)] + [(\delta_{n+1} - \delta_n - \gamma_{n+1} + \gamma_n) (x - c_n) - \\
- & (d_{n+1} - d_n - c_{n+1} + c_n) (y - \gamma_n) + (d_n - c_n) (\gamma_{n+1} - \gamma_n)] \cdot [(\beta_n - \alpha_n) (x - a_n) - \\
- & (b_n - a_n) (y - \alpha_n)] + \lambda_{n+1} J_1 (x, y) + \nu_{n+1} J_2 (x, y) = 0.
\end{align*}
\]

(12)

\[
[(\gamma_n - \alpha_n) (x - a_n) - (c_n - a_n) (y - \alpha_n)] \cdot [(\delta_n - \beta_n) (x - b_n) - \\
- (d_n - b_n) (y - \beta_n)] + [(\gamma_{n+1} - \gamma_n - \alpha_{n+1} + \alpha_n) (x - a_n) - \\
- (\gamma_n - \alpha_n) (a_{n+1} - a_n) - (c_{n+1} - c_n - a_{n+1} + a_n) (y - \alpha_n) + \\
+ (c_n - a_n) (\alpha_{n+1} - \alpha_n)] \cdot [(\delta_n - \beta_n) (x - b_n) - (d_n - b_n) (y - \beta_n)] + \\
+ [(\delta_{n+1} - \delta_n - \beta_{n+1} + \beta_n) (x - b_n) - (\delta_n - \beta_n) (b_{n+1} - b_n) - \\
- (d_{n+1} - d_n - b_{n+1} + b_n) (y - \beta_n) + \\
+ (d_n - b_n) (\beta_{n+1} - \beta_n)] \cdot [(\gamma_n - \beta_n) (x - a_n) - \\
- (c_n - a_n) (y - \alpha_n)] + \varphi_{n+1} J_1 (x, y) + \varphi_{n+1} J_2 (x, y) = 0.
\]

(13)

Equalities (12) and (13) represent polynomial identities. These identities facilitate to determine

\[a_{n+1}, \alpha_{n+1}, b_{n+1}, \beta_{n+1}, \ldots, \lambda_{n+1}, \nu_{n+1}, \varphi_{n+1}
\]

as the functions of

\[a_n, \alpha_n, b_n, \beta_n, c_n, \gamma_n, d_n, \delta_n.
\]

This determination shall be accomplished in such a way that in equalities (12) and (13) instead of (x, y) we replace (a_n, \alpha_n), (b_n, \beta_n), (c_n, \gamma_n), (d_n, \delta_n), after that, following eight equations are obtained

\[
(\alpha_n - \beta_n) a_{n+1} + (b_n - a_n) \alpha_{n+1} + \frac{J_1 (a_n, \alpha_n) \lambda_{n+1} + J_2 (a_n, \alpha_n) \nu_{n+1}}{(\delta_n - \gamma_n) (a_n - c_n) - (d_n - c_n) (\alpha_n - \gamma_n)} = a_n (\alpha_n - \beta_n) + \alpha_n (b_n - a_n),
\]

(14)

\[
(\alpha_n - \gamma_n) a_{n+1} + (c_n - a_n) \alpha_{n+1} + \frac{J_1 (a_n, \alpha_n) \varphi_{n+1} + J_2 (a_n, \alpha_n) \varphi_{n+1}}{(\delta_n - \beta_n) (a_n - b_n) - (d_n - b_n) (\alpha_n - \beta_n)} = a_n (\alpha_n - \gamma_n) + \alpha_n (c_n - a_n),
\]

(15)

\[
(\alpha_n - \beta_n) b_{n+1} + (b_n - a_n) \beta_{n+1} + \frac{J_1 (b_n, \beta_n) \lambda_{n+1} + J_2 (b_n, \beta_n) \nu_{n+1}}{(\delta_n - \gamma_n) (b_n - c_n) - (d_n - c_n) (\beta_n - \gamma_n)} = b_n (\alpha_n - \beta_n) + \beta_n (b_n - a_n),
\]

(16)

\[
(\beta_n - \delta_n) b_{n+1} + (d_n - b_n) \beta_{n+1} + \frac{J_1 (b_n, \beta_n) \varphi_{n+1} + J_2 (b_n, \beta_n) \varphi_{n+1}}{(\gamma_n - \alpha_n) (b_n - a_n) - (c_n - a_n) (\beta_n - \alpha_n)} = b_n (\beta_n - \delta_n) + \beta_n (d_n - b_n),
\]

(17)
\begin{align}
(18) \quad (\gamma_n - \delta_n) c_{n+1} + (d_n - c_n) \gamma_{n+1} + & \quad \frac{J_1(c_n, \gamma_n) \lambda_{n+1} + J_2(c_n, \gamma_n) \nu_{n+1}}{(\beta_n - \alpha_n) (c_n - a_n) - (b_n - a_n) (\gamma_n - \alpha_n)} \\
& = c_n (\gamma_n - \delta_n) + \gamma_n (d_n - c_n),
\end{align}

\begin{align}
(19) \quad (\alpha_n - \gamma_n) c_{n+1} + (c_n - a_n) \gamma_{n+1} + & \quad \frac{J_1(c_n, \gamma_n) \rho_{n+1} + J_2(c_n, \gamma_n) \varphi_{n+1}}{(\delta_n - \beta_n) (c_n - b_n) - (d_n - b_n) (\gamma_n - \beta_n)} \\
& = c_n (\alpha_n - \gamma_n) + \gamma_n (c_n - a_n),
\end{align}

\begin{align}
(20) \quad (\gamma_n - \delta_n) d_{n+1} + (d_n - c_n) \delta_{n+1} + & \quad \frac{J_1(d_n, \delta_n) \lambda_{n+1} + J_2(d_n, \delta_n) \nu_{n+1}}{(\beta_n - \alpha_n) (d_n - a_n) - (b_n - a_n) (\delta_n - \alpha_n)} \\
& = d_n (\gamma_n - \delta_n) + \delta_n (d_n - c_n),
\end{align}

\begin{align}
(21) \quad (\beta_n - \delta_n) d_{n+1} + (d_n - b_n) \delta_{n+1} + & \quad \frac{J_1(d_n, \delta_n) \rho_{n+1} + J_2(d_n, \delta_n) \varphi_{n+1}}{(\alpha_n - \gamma_n) (d_n - a_n) - (c_n - a_n) (\delta_n - \alpha_n)} \\
& = d_n (\beta_n - \delta_n) + \delta_n (d_n - b_n).
\end{align}

If we designate, further, on the basis of given schema, with \( L_n \) and \( M_n \) average points of straight lines \( A_n B_n \) and \( C_n D_n \), and \( A_n C_n \) and \( B_n D_n \) respectively, coordinates of these points are given by formulae
\[
\begin{align*}
& \quad l_n = \frac{(d_n - c_n) [ (\alpha_n - \gamma_n) (b_n - a_n) - a_n (\beta_n - \alpha_n) ] + c_n (\delta_n - \gamma_n) (b_n - a_n)}{(b_n - a_n) (\delta_n - \gamma_n) - (d_n - c_n) (\beta_n - \alpha_n)}, \\
& \quad \theta_n = \frac{(\beta_n - \alpha_n) [ (c_n (\delta_n - \gamma_n) - \gamma_n (d_n - c_n)) + (\delta_n - \gamma_n) [ (\alpha_n (b_n - a_n) - a_n (\beta_n - \alpha_n)) ] }{(b_n - a_n) (\delta_n - \gamma_n) - (d_n - c_n) (\beta_n - \alpha_n)}, \\
& \quad m_n = \frac{(d_n - b_n) [ (\alpha_n - \beta_n) (c_n - a_n) - a_n (\gamma_n - \alpha_n) ] + b_n (\delta_n - \beta_n) (c_n - a_n)}{(c_n - a_n) (\delta_n - \beta_n) - (d_n - b_n) (\gamma_n - \alpha_n)}, \\
& \quad \psi_n = \frac{(\gamma_n - \alpha_n) [ b_n (\delta_n - \beta_n) - \beta_n (d_n - b_n)] + (\delta_n - \beta_n) [ (\alpha_n (c_n - a_n) - a_n (\gamma_n - \alpha_n)] }{(c_n - a_n) (\delta_n - \beta_n) - (d_n - b_n) (\gamma_n - \alpha_n)}.
\end{align*}
\]

Substituting in basic formulae (12) and (13) values for \( (x, y) \) with calculated values \( (l_n, \theta) \) and \( (m_n, \psi) \) we obtain similarly to equations (14)—(21), four equations more
\begin{align}
(22) \quad J_1(l_n, \theta_n) \lambda_{n+1} + J_2(l_n, \theta_n) \nu_{n+1} = 0,
\end{align}
(because the point $L_n(\gamma_n, \theta)$ lays in the intersection of straight lines $A_nB_n$ and $C_nD_n$).

\[
[(\theta_n - \gamma_n) a_{n+1} + (c_n - l_n) x_{n+1} + (\gamma_n - \theta_n) c_{n+1} + \\
+ (l_n - a_n) \gamma_{n+1} \cdot [(\delta_n - \beta_n) (l_n - b_n) - (d_n - b_n) (\theta_n - \beta_n)] + \\
+ [(\theta_n - \delta_n) b_{n+1} + (d_n - l_n) \beta_{n+1} + (\beta_n - \theta_n) d_{n+1} + \\
+ (l_n - b_n) \delta_{n+1}] \cdot [(\gamma_n - \alpha_n) (l_n - a_n) - (c_n - a_n) (\theta_n - \alpha_n)] + \\
+ J_1(l_n, \theta_n) \varphi_{n+1} + J_2(l_n, \theta_n) \varphi_{n+1} = [(\gamma_n - \alpha_n) (l_n - a_n) - \\
- (c_n - a_n) (\theta_n - \alpha_n)] \cdot [(d_n - b_n) (\theta_n - \beta_n) - (\delta_n - \beta_n) (l_n - b_n)] + \\
+ [a_n (\theta_n - \gamma_n) + a_n (c_n - l_n) + c_n (\alpha_n - \theta_n) + \\
+ \gamma_n (l_n - a_n)] \cdot [(\delta_n - \beta_n) (l_n - b_n) - (d_n - b_n) (\theta_n - \beta_n)] + \\
+ [b_n (\theta_n - \delta_n) + \beta_n (d_n - l_n) + d_n (\beta_n - \theta_n) + \delta_n (l_n - b_n)] \times \\
\times [(\gamma_n - \alpha_n) (l_n - a_n) - (c_n - a_n) (\theta_n - \alpha_n)],
\]

(23)

\[
[(\psi_n - \beta_n) a_{n+1} + (b_n - m_n) x_{n+1} + (\alpha_n - \psi_n) b_{n+1} + \\
+ (m_n - a_n) \beta_{n+1} \cdot [(\delta_n - \gamma_n) (m_n - c_n) - (d_n - c_n) (\psi_n - \gamma_n)] + \\
+ [(\psi_n - \delta_n) c_{n+1} + (d_n - m_n) \gamma_{n+1} + (\gamma_n - \psi_n) d_{n+1} + \\
+ (m_n - c_n) \delta_{n+1}] \cdot [(\beta_n - \alpha_n) (m_n - a_n) - (b_n - a_n) (\psi_n - \alpha_n)] + \\
+ J_1(m_n, \psi_n) \lambda_{n+1} + J_2(m_n, \psi_n) \mu_{n+1} = \\
= [(\beta_n - \alpha_n) (m_n - a_n) - (b_n - a_n) (\psi_n - \alpha_n)] \cdot [(d_n - c_n) (\psi_n - \gamma_n) - \\
- (\delta_n - \gamma_n) (m_n - c_n)] + [a_n (\psi_n - \beta_n) + \alpha_n (b_n - m_n) + b_n (\alpha_n - \psi_n) + \\
+ \beta_n (m_n - a_n)] \cdot [(\delta_n - \gamma_n) (m_n - c_n) - (d_n - c_n) (\psi_n - \gamma_n)] + \\
+ [c_n (\psi_n - \delta_n) + \gamma_n (d_n - m_n) + d_n (\gamma_n - \psi_n) + \\
+ \delta_n (m_n - c_n)] \cdot [(\beta_n - \alpha_n) (m_n - a_n) - (b_n - a_n) (\psi_n - \alpha_n)],
\]

(24)

\[
J_1(m_n, \psi_n) \varphi_{n+1} + J_2(m_n, \psi_n) \varphi_{n+1} = 0,
\]

(25)

(because the point $M_n(m_n, \psi_n)$ lays in the intersection of straight lines $A_nC_n$ and $B_nD_n$).

System of equations (14) — (25) represents a system of linear algebraic equations which aids the determination of series

\[a_{n+1}, x_{n+1}, b_{n+1}, \beta_{n+1}, \ldots, \lambda_{n+1}, \mu_{n+1}, \varphi_{n+1}, \varphi_{n+1}\]

in function of series

\[a_n, x_n, a_n, \beta_n, c_n, \gamma_n, d_n, \delta_n, \bar{\lambda}, \bar{\mu}, \bar{\varphi}, \bar{\varphi}\]

in relation to these series following assertion is valuable:

If the series $a_n, x_n, b_n, \beta_n, c_n, \gamma_n, d_n, \delta_n, \lambda_n, \mu_n, \varphi_n, \varphi_n$ converges one after another towards $a, x, b, \beta, c, \gamma, d, \delta, \lambda, \mu, \varphi, \varphi$ and if $\lambda \varphi - \varphi \mu \neq 0$, then the points $A(a, \alpha), B(b, \beta), C(c, \gamma), D(d, \delta)$ determine all solutions of system (8).
Proof.

Equalities (12) and (13) in limit case give
\[ \left[(\beta - \alpha) (x - a) - (b - a) (y - \alpha)\right] \cdot \left[(\delta - \gamma) (x - c) - (d - c) (y - \gamma)\right] + \lambda J_1 (x, y) + \mu J_2 (x, y) = 0, \]
\[ \left[(\gamma - \alpha) (x - a) - (c - a) (y - \alpha)\right] \cdot \left[(\delta - \beta) (x - b) - (d - b) (y - \beta)\right] + \rho J_1 (x, y) + \varphi J_2 (x, y) = 0. \]

On the basis of previous equalities taking into account the assumption that \( \lambda \varphi - \rho \mu \neq 0 \), it may be concluded that the system \( J_1 (x, y) = 0, J_2 (x, y) = 0 \) is equivalent to the system
\[ \left[(\beta - \alpha) (x - a) - (b - a) (y - \alpha)\right] \cdot \left[(\delta - \gamma) (x - c) - (d - c) (y - \gamma)\right] = 0, \]
\[ \left[(\gamma - \alpha) (x - a) - (c - a) (y - \alpha)\right] \cdot \left[(\delta - \beta) (x - b) - (d - b) (y - \beta)\right] = 0. \]
The proof is finished, as all solutions of that system are the points \( A (a, \alpha), B (b, \beta), C (c, \gamma), D (d, \delta) \).

Note. On the basis of investigations with are not completely finished it may be assumed that the convergence is quadratic.

Application

Progressivity of application of exposed algorithm may be seen from the following: On the basis of given starting data
\[ (a_0, \alpha_0), (b_0, \beta_0), (c_0, \gamma_0), (d_0, \delta_0) \]
by use of formula for \( l, \theta, m, \psi \), we first calculate the values \( (l_0, \theta_0), (m_0, \psi_0) \). In this way a complete group of data is formed, necessary for calculation — on the basis of the system of linear algebraic equations (14)—(25) of values \( (a_1, \alpha_1), (b_1, \beta_1), (c_1, \gamma_1), (d_1, \delta_1), \lambda_1, \mu_1, \varphi_1 \) and they, after calculations by using formulae for \( l, \theta, m, \psi \) of values \( (l_1, \theta), (m_1, \psi) \), represent first corrections of initial conditions. Further procedure continue in the same way, by calculating \( (a_2, \alpha_2), (b_2, \beta_2), (c_2, \gamma_2), (d_2, \delta_2), \lambda_2, \mu_2, \varphi_2 \), until results of desired precisions are obtained.

This procedure will be illustrated by following example.

A system of equations is given
\[ J_1 (x, y) \equiv x^2 - 4xy + 2y^2 - x - 2y = 0, \]
\[ J_2 (x, y) \equiv 3x^2 - 14xy + 2y^2 - 3x + 8y = 0, \]
its solutions are known
\[ A (a; \alpha) = A (1; 3), \quad B (b; \beta) = B (5; 1), \]
\[ C (c; \gamma) = C (0; 0), \quad D (d; \delta) = D (1; 0). \]

By the application of exposed algorithm it is necessary to determine solutions of the given system, under assumption that they are unknown, starting with following initial conditions
\[ A_0 (a_0; \alpha_0) = A_0 (2; 4), \quad B_0 (b_0; \beta_0) = B_0 (3; 2), \]
\[ C_0 (c_0; \gamma_0) = C_0 (-1; -1), \quad D_0 (d_0; \delta_0) = D_0 (0,5; -1). \]
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<th>3</th>
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</table>

An algorithm for the solution of a system of nonlinear algebraic equations.
Results of calculations obtained on digital computer CII 10070 are given in following table.

As may be seen from the data indicated in the table using these iterative approximations, solutions with precision on fifth digit are obtained, with total working time of CII 10070 computer of 1.79 min.

The same example was solved with other initial conditions

\[ A_0(a_0; \alpha_0) = A_0(0; 12), \quad B_0(b_0; \beta_0) = B_0(3; 10) \]
\[ C_0(c_0; \gamma_0) = C_0(-4; -2), \quad D_0(d_0; \delta_0) = D_0(0,1; -3) \]

and the same solutions were obtained after twelve iterations, total computer working time being 1.92 min.

Results of these calculations were obtained in the Laboratory for Applied Mathematics of the MTI, Beograd. We thank S. Stamatović and M. Jovanović for complaisance and help they extended to us.

REFERENCES
