A COMPLETENESS THEOREM FOR ONE CLASS OF PROPOSITIONAL CALCULI

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Let $L = \{p_i \mid i \in I\}$ be a given set of so called propositional letters and $O = \{o_j \mid j \in J\}$ another set of operation symbols. Associated with each operation symbol $o_j$ there is a non-negative integer $l(o_j)$ called the length of $o_j$. We denote by $C$ the set of all operation symbols having the length zero. The elements of $C$ we call constant symbols, and the other elements of $O$ are called connective symbols. For the sets $L$ and $O$ we suppose:

$$L \cap O = \emptyset, \quad L \cup C \neq \emptyset, \quad O \setminus C \neq \emptyset.$$ 

Thus, the set $L$ can be the empty set. From the elements of the sets $L$ and $O$ we define formulae in the usual way. Namely, the set $\mathcal{F}$ of all formulae is the smallest set (a subset of the set of all strings built up from the members of $L \cup O$) which satisfies the following conditions

(1) (i) $L \subseteq \mathcal{F}$

(ii) $C \subseteq \mathcal{F}$

(iii) If $\varphi_1, \ldots, \varphi_n \in \mathcal{F}$ and $l(o_j) = n$, then $o_j \varphi_1 \cdots \varphi_n \in \mathcal{F}$

The signs $P_1, P_2, P_3, \ldots$ are used to denote arbitrary formulae (i.e. as variables for formulae). From these symbols and the elements of $O$ formula schemes are built up. Namely, if in the above definition of formulae the part (i) is replaced by $\{P_1, P_2, \ldots\} \subseteq \mathcal{F}$, we obtain the definition of the set all formula schemes. Further, let $Ax$ be a nonempty set of formula schemes, and $R$ a set of certain rule-schemes of inference, each of the form

(2) $\frac{\Phi_1, \ldots, \Phi_n}{\Psi}$ \quad ($n \geq 1$)

where $\Phi_1, \ldots, \Phi_n, \Psi$ are formula schemes.
By the sets $Ax$ and $R$ is defined, a so called propositional calculus. More precisely a propositional calculus may be defined as the ordered pair $(Ax, R)$. Let

$$(3) \quad \varphi(P_1, P_2, \ldots, P_k)$$

be any element of $Ax$, where all the elements of the set $\{P_1, P_2, \ldots\}$ occurring in $(3)$ are among $P_1, P_2, \ldots, P_k$. If $\varphi_1, \varphi_2, \ldots, \varphi_k$ are any formulae, then the string

$$(4) \quad \varphi(\varphi_1, \varphi_2, \ldots, \varphi_k)$$

is a formula as well. The formulae as (4), for which we also say that they are of the form (3), we call axioms (of the calculus $(Ax, R)$). By $Ax$ we denote the set of all axioms. According to the given definition of the sets of formulae and formula schemes the following statement is true:

$$(5) \quad \text{In any axiom its propositional letters may be replaced by any formulae. In such a way from an axiom we again obtain an axiom.}$$

Let $\mathcal{P}$ i.e. $(Ax, R)$ be a propositional calculus. By $\text{Th} (\mathcal{P})$ we denote the set of all its theorems, and by $\vdash F$ we denote the fact that the formula $F$ is a theorem (of $\mathcal{P}$).

Now we define a model for $\mathcal{P}$. It is a usual definition. Namely, let $M$ be a nonempty set, and let 1 denote one of its elements. Further, with each $o_j$ we associate an object $\delta_j$ in the following way:

If $o_j \subseteq C$, then $\delta_j$ is a certain chosen element of $M$.

If $o_j \subseteq O \setminus C$, and $l(o_j) = n$, then $\delta_j$ is an $n$-ary operation of $M$.

By the set $M$ and the operations $\delta_j$ an algebra $M$ is determined. The set $M$ is its domain. On certain conditions we say that an algebra $M$ is a model for a propositional calculus. Namely, that is exactly in the case when:

$$(6) \quad \text{Each axiom } f \text{ is valid in the algebra } M, \text{ i.e. the equality } \bar{f} = 1 \text{ is satisfied for all possible substitutions of the elements of } M \text{ for the letters } p_i \text{ (occurring in the formula } f). \text{ By } \bar{f} \text{ we denote the string obtained from } f \text{ by replacing: } o_j \text{ by } \delta_j.$$ 

$$(7) \quad \text{The rule schemes of } \mathcal{P} \text{ are satisfied by } M \text{ in the sense that validity is preserved by them; i.e. if (2) is any rule schema then the implication}$$

$$\text{If } \overline{\Phi_1} = 1, \ldots, \overline{\Phi_n} = 1, \text{ then } \overline{\Psi} = 1$$

is true for all $P_1, P_2, \ldots \subseteq M$.

From the conditions (6), (7), it is immediate that each member of the set $\text{Th}(\mathcal{P})$, i.e. any theorem of $\mathcal{P}$, is also valid in the model $M$. It may happen that for a certain algebra $M$ we have (Completeness theorem):

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1) Therefore the propositional letters may be called variables.

2) If $l(o) = 0$, then $\delta_j$ is called a operation of the length zero. The element 1 is a operation of such kind,
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(8) Any formula \( f \) of the propositional calculus \( \mathcal{P} \) is a theorem if and only if \( f \) is valid in the algebra \( M \).

In such a case we say that the algebra \( M \) is an adequate model for \( \mathcal{P} \).

In this paper we give a necessary and sufficient condition for \( \mathcal{P} \) to have an adequate model.

In order to express this condition, we define a new formal theory \( \mathcal{P}(=, 1) \) in the following way.

Denote by \( O_1 \) the set \( O \cup \{1\} \) and let by definition 1 be \( C_0 \) (provided \( C_0 \subseteq O \)). By definition (1) we obtain the so called terms of \( \mathcal{P}(=, 1) \).

The formulae of \( \mathcal{P}(=, 1) \) are strings of the form \( t_1 = t_2 \), where \( t_1, t_2 \) are terms.

The formula schemes of \( \mathcal{P}(=, 1) \) are, shortly, strings which can be obtained from the formulae of \( \mathcal{P}(=, 1) \) by all possible replacements of elements of the set \( L \) by elements of the set \( \{P_1, P_2, \ldots, P_n, \ldots\} \).

The axiom schemes of \( \mathcal{P}(=, 1) \) are:

(9) all formula schemes of the form \( f = 1 \), where \( f \) is any axiom schema of \( \mathcal{P} \) (provided that 1 is included as a member of \( C \))

(10) all formula schemes of the form \( t = t \).

The rule schemes of \( \mathcal{P}(=, 1) \) are:

(11) \( \varphi_1 = 1, \ldots, \varphi_n = 1 \), whenever \( \Phi \), \( \psi \) is a rule schema of \( \mathcal{P} \).

(12) \( t_1 = t_2 \), \( t_1 = t_2, t_2 = t_3 \), \( t_1 = t_1', \ldots, t_n = t_n' \)

\( t_2 = t_1 \), \( t_1 = t_3 \), \( o_j t_1 \cdots t_n = o_j t_1' \cdots t_n' \)

where \( t_1, t_2, \ldots \) are any terms, \( o_j \) any \( n \)-ary operation symbol.

We also note that:

(13) In the case of the theory \( \mathcal{P}(=, 1) \) it is supposed that \( P_1, P_2, \ldots \) may be replaced by any terms. In such a way, for example, from axiom-schemes of \( \mathcal{P}(=, 1) \) we get the axioms of \( \mathcal{P}(=, 1) \).

There is an interesting connection between the theories \( \mathcal{P}, \mathcal{P}(=, 1) \):

(14) An algebra is a model for the calculus \( \mathcal{P} \) if and only if \( M \) is a model\(^3\) for the theory \( \mathcal{P}(=, 1) \).

This immediately follows from the definitions of models for \( \mathcal{P}(=, 1) \), and from (10), (12).

Further, we note that the theory \( \mathcal{P}(=, 1) \) always has at least one adequate algebra. Namely, these are the so called free algebras. One of such algebras \( \Phi \) can be described in the following way.

\(^3\) As a matter of fact, its normal model, i.e. a model in which \( = \) is interpreted as the equality,
Let $T$ be the set of all terms. On the set $T$ define the relation $\sim$ as follows:

\begin{equation}
(15) \quad t_1 \sim t_2 \text{ if and only if } \quad \vdash \quad t_1 = t_2 \quad \text{ in } \mathcal{P}(-, 1)
\end{equation}

i.e. $t_1 \sim t_2$ if and only if $t_2$ can be obtained from $t_1$ using the axioms of the forms (9), (10) and the rules (11), (12). The relation $\sim$ is an equivalence relation. This follows from the axiom of the form $t = t$, and the rules

\[
\begin{align*}
\frac{t_1 = t_2}{t_2 = t_1}, & \quad \frac{t_1 = t_2, t_2 = t_3}{t_1 = t_3}
\end{align*}
\]

The members of $T/\sim$ i.e. of the quotient set will be denoted by $C_t$, where $t \in T$. These classes are the only elements of the algebra $\Phi$. Its operations $\delta_j$ are defined as follows:

If $l(o_j) = 0$, then $\delta_j \overset{\text{def}}{=} C_{o_j}$. So $\overline{1} = C_1$.

If $l(o_j) = n (> 0)$, then $\delta_j$ is the $n$-ary operation determined by

\begin{equation}
(16) \quad \delta_j(C_{t_1}, \ldots, C_{t_n}) = C_{o_j t_1 \ldots t_n}
\end{equation}

In the definition (16) the result, i.e. $\delta_j(C_{t_1}, \ldots, C_{t_n})$ is defined by $t_1, \ldots, t_n$, i.e. by certain members of the classes $C_{t_1}, \ldots, C_{t_n}$ respectively. Logical correctness of this definition follows from the rule schema

\[
\begin{align*}
& t_1 = t'_1, \ldots, t_n = t'_n \\
& o_j t_1 \cdots t_n = o_j t'_1 \cdots t'_n
\end{align*}
\]

The set $T/\sim$ and the operation $\delta_j$ determine the algebra $\Phi$.

The equality (16) can be generalized in the following way. By

\[
t(a_1, \ldots, a_n, o_{j_1}, \ldots, o_{j_m})
\]

denote a term, where $a_1, \ldots, a_n$ as well as $o_{j_1}, \ldots, o_{j_m}$ are all elements of the set $L$ i.e. of the set $O_1$, occurring in the term. Then by induction on $m$ the following equality may easily be proved

\begin{equation}
(17) \quad t(C_{a_1}, \ldots, C_{a_n}, \delta_{j_1}, \ldots, \delta_{j_m}) = C_t(a_1, \ldots, a_n, o_{j_1}, \ldots, o_{j_m})
\end{equation}

We now give a proof that the algebra $\Phi$ is an adequate model of the theory $\mathcal{P}(-, 1)$.

The formula

\[
t_1(a_1, \ldots, a_n, o_{t_1}, \ldots, o_{j_{m_1}}) = t_2(b_1, \ldots, b_{n_2}, o_{k_1}, o_{k_2}, \ldots, o_{k_{m_2}})
\]

is valid in $\Phi$. 
if and only if the equality
\[ t_1(C_{x_1}, \ldots, C^{\overline{n_1}}, \delta_j, \ldots, \delta_{j_1}) = t_2(C_{y_1}, \ldots, C^{\overline{n_2}}, \delta_k, \ldots, \delta_{k_1}) \]
holds in \( \Phi \) for all \( x_1, \ldots, x_{n_1}, y_1, \ldots, y_{n_2} \in T \).

if and only if the equality
\[ C_{t_1}(x_1, \ldots, x_{n_1}, o_{j_1}, \ldots, o_{j_1}) = C_{t_2}(y_1, \ldots, y_{n_2}, o_k, \ldots, o_{k_1}) \]
holds in \( \Phi \) for all \( x_1, \ldots, x_{n_1}, y_1, \ldots, y_{n_2} \in T \).

if and only if
\[ t_1(x_1, \ldots, x_{n_1}, o_{j_1}, \ldots, o_{j_1}) \sim t_2(y_1, \ldots, y_{n_2}, o_k, \ldots, o_{k_1}) \]
for all \( x_1, \ldots, x_{n_1}, y_1, \ldots, y_{n_2} \in T \).

if and only if
\[ \vdash t_1(x_1, \ldots, x_{n_1}, o_{j_1}, \ldots, o_{j_1}) = t_2(y_1, \ldots, y_{n_2}, o_k, \ldots, o_{k_1}) \]
\( \mathcal{P}(=, 1) \)
for all \( x_1, \ldots, x_{n_1}, y_1, \ldots, y_{n_2} \in T \).

if and only if
\[ \vdash t_1(a_1, \ldots, a_{n_1}, o_{j_1}, \ldots, o_{j_1}) = t_2(b_1, \ldots, b_{n_2}, o_k, \ldots, o_{k_1}) \]
\( \mathcal{P}(=, 1) \)

The last step in the proof is based on the fact that in any theorem of \( \mathcal{P}(=, 1) \), say \( \varphi(a_1, a_2, \ldots) \), where \( a_1, a_2, \ldots \in L \), the propositional letters \( a_1, a_2, \ldots \) may be replaced by any terms (see (13)).
Now we prove our main result.

**Theorem.** For a propositional calculus \( \mathcal{P} \) there is an adequate algebra if and only if the theory \( \mathcal{P}(=, 1) \) is not a creative extension of \( \mathcal{P} \), i.e. for every formula \( f \) of \( \mathcal{P} \) the equivalence\(^a\) is true

\[
\vdash f \quad \text{if and only if} \quad \vdash f = 1 \quad \text{in} \quad \mathcal{P}(=, 1)
\]

\(^a\) The implication

\[ \vdash f, \quad \text{then} \quad \vdash f = 1 \quad \text{in} \quad \mathcal{P}(=, 1) \]

is true. Therefore the equivalence (18) may be reduced to the following condition

There is no formula \( f \) of \( \mathcal{P} \) such that

\[ \vdash f = 1, \text{ but not } \vdash f \quad \text{in} \quad \mathcal{P}(=, 1) \]
Proof. First, suppose that $\mathcal{P} (=, 1)$ is a creative extension of $\mathcal{P}$. Then there is a formula $f$ of $\mathcal{P}$ such that:

$$\vdash f = 1 \text{ and not } \vdash f \quad \mathcal{P} (=, 1)$$

Let us also suppose that the calculus $\mathcal{P}$ has an adequate algebra, say $M$. This algebra is a model of the theory $\mathcal{P} (=, 1)$ as well. Therefore, it follows:

$$f = 1 \text{ is valid in } M$$

Hence, we infer that $\vdash f$. But from $\vdash f$ and not $\vdash f$ we conclude that the implication

$\mathcal{P}$ has an adequate model

$$\Rightarrow \mathcal{P} (=, 1) \text{ is not a creative extension of } \mathcal{P}$$

is true.

Second, suppose that $\mathcal{P} (=, 1)$ is not a creative extension of $\mathcal{P}$. According to the given proof there is an algebra $\Phi$ being an adequate algebra of $\mathcal{P} (=1)$. But this algebra is adequate for $\mathcal{P}$ as well, which follows from the proof:

Let $f(a_1, \ldots, o_{j_1}, \ldots)$ be any formula of $\mathcal{P}$. Then:

$$f(a_1, \ldots, o_{j_1}, \ldots) \text{ is valid in } \Phi$$

if and only if the equality

$$f(C_{x_1}, \ldots, o_{j_1}, \ldots) = C_1$$

holds in $\Phi$ for all $x_1, \ldots \in T$ if and only if the equality

$$C_f(x_1, \ldots, o_{j_1}, \ldots) = C_1$$

holds in $\Phi$ for all $x_1, \ldots \in T$ if and only if

$$\vdash f(x_1, \ldots, o_{j_1}, \ldots) = 1 \quad \mathcal{P} (=, 1)$$

hold in $\Phi$ for all $x_1, \ldots \in T$ if and only if

$$\vdash f(a_1, \ldots, o_{j_1}, \ldots) = 1 \quad \mathcal{P} (=, 1)$$

if and only if

$$\vdash f(a_1, \ldots, o_{j_1}, \ldots)$$