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A COMPLETENESS THEOREM FOR ONE CLASS OF PROPOSITIONAL CALCULI

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Let $L=\{p_i \mid i\in I\}$ be *a* given set of so called *propositional letters* and $O=\{o_j \mid j\in J\}$ another set of *operation symbols*. Associated with each operation symbol o_j there is a non-negative integer $l(o_j)$ called *the length of* o_j. We denote by C the set of all operation symbols having the length zero. The elements of C we call *constant symbols*, and the other elements of O are called *connective symbols*. For the sets L and O we suppose:

$$L\cap O=\varnothing, \qquad L\cup C\neq\varnothing, \qquad O\setminus C\neq\varnothing .$$

Thus, the set L can be the empty set. From the elements of the sets L and O we define *formulae* in the usual way. Namely, the set $\mathcal{F}$ of all formulae is the smallest set (a subset of the set of all strings built up from the members of $L\cup O$) which satisfies the following conditions

(1) (i) $L\subseteq\mathcal{F}$

(ii) $C\subseteq\mathcal{F}$

(iii) *If* $\varphi_1, \ldots, \varphi_n\in\mathcal{F}$ and $l(o_j)=n$, then $o_j\varphi_1\cdots\varphi_n\in\mathcal{F}$

The signs $P_1, P_2, P_3, \ldots$ are used to denote arbitrary formulae (i.e. as variables for formulae). From these symbols and the elements of O *formula schemes* are built up. Namely, if in the above definition of formulae the part (i) is replaced by $\{P_1, P_2, \ldots\}\subseteq\mathcal{F}$, we obtain the definition of the set all formula schemes. Further, let Ax be a nonempty set of formula schemes, and R a set of certain rule-schemes of inference, each of the form

$$\frac{\Phi_1, \ldots, \Phi_n}{\Psi} \qquad (n\geqslant 1) \tag{2}$$

where $\Phi_1, \ldots, \Phi_n, \Psi$ are formula schemes.

By the sets Ax and R is defined, a so called *propositional calculus*. More precisely a propositional calculus may be defined as the ordered pair (Ax, R). Let

$$(3) \qquad \varphi(P_1, P_2, \ldots, P_k)$$

be any element of Ax, where all the elements of the set $\{P_1, P_2, \ldots\}$ occurring in (3) are among $P_1, P_2, \ldots, P_k$. If $\varphi_1, \varphi_2, \ldots, \varphi_k$ are any formulae, then the string

$$(4) \qquad \varphi(\varphi_1, \varphi_2, \ldots, \varphi_k)$$

is a formula as well. The formulae as (4), for which we also say that they are of the form (3), we call *axioms* (of the calculus (Ax, R). By Ax we denote the set of all axioms. According to the given definition of the sets of formulae and formula schemes the following statement is true:

(5) *In any axiom its propositititional letters may by replaced by any formulae. In such a way from an axiom we again obtain an axiom.*

Let $\mathcal{P}$ i.e. (Ax, R) be a propositional calculus. By Th $(\mathcal{P})$ we denote the set of all its *theorems*, and by $\underset{\mathcal{P}}{\vdash} F$ we denote the fact that the formula F is a theorem (of $\mathcal{P}$).

Now we define *a model for* $\mathcal{P}$. It is a usual definition. Namely, let M be a nonempty set, and let 1 denote one of its elements. Further, with each o_j we associate an object $\bar{o}_j$ in the following way:

If $o_j \in C$, then $\bar{o}_j$ is a certain chosen element of M.

If $o_j \in O \setminus C$, and $l(o_j) = n$, then $\bar{o}_j$ is an n-ary operation of M.

By the set M and the operations[2] $\bar{o}_j$ an *algebra* M is determined. The set M is its *domain*. On certain conditions we say that an algebra M is a model for a propositional calculus. Namely, that is exactly in the case when:

(6) Each axiom f is valid in the algebra M, i.e. the equality $\bar{f} = 1$ is satisfied for all possible substitutions of the elements of M for the letters p_i (occurring in the formula f). By $\bar{f}$ we denote the string obtained from f by replacing: o_j by $\bar{o}_j$.

(7) The rule schemes of $\mathcal{P}$ are satisfied by M in the sense that validity is preserved by them; i.e. if (2) is any rule schema then the implication

$$\text{If } \bar{\Phi}_1 = 1, \ldots, \bar{\Phi}_n = 1, \text{ then } \bar{\Psi} = 1$$

is true for all $P_1, P_2, \ldots \in M$.

From the conditions (6), (7), it is immediate that each member of the set $Th(\mathcal{P})$, i.e. any theorem of $\mathcal{P}$, is also valid in the model M. It may happen that for a certain algebra M we have (*Completeness theorem*):

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[1] Therefore the propositional letters may be called *variables*.

[2] If $l(o_j) = 0$, then $\bar{o}_j$ is called a operation of the length zero. The element 1 is a operation of such kind.

(8) *Any formula f of the propositional calculus $\mathcal{P}$ is a theorem if and only if f is valid in the algebra M.*

In such a case we say that the algebra M is an *adequate model* for $\mathcal{P}$.

In this paper we give a necessary and sufficient condition for $\mathcal{P}$ to have an edequate model.

In order to express this condition, we define a new formal theory $\mathcal{P}(=, 1)$ in the following way.

Denote by O_1 the set $O \cup \{1\}$ and let by definition 1 be C_0 (provided $C_0 \notin O$). By definition (1) we obtain the so called *terms* of $\mathcal{P}(=, 1)$.

The *formulae* of $\mathcal{P}(=, 1)$ are strings of the form $t_1 = t_2$, where t_1, t_2 are terms.

The *formula schemes* of $\mathcal{P}(=, 1)$ are, shortly, strings which can be obtained from the formulae of $\mathcal{P}(=, 1)$ by all posible replacements of elements of the set L by elements of the set $\{P_1, P_2, \ldots, P_n, \ldots\}$.

The *axiom schemes* of $\mathcal{P}(=, 1)$ are:

(9) all formula schemes of the form $f = 1$, where f is any axiom schema of $\mathcal{P}$ (provided that 1 is included as a member of C)

(10) all formula schemes of the form $t = t$.

The *rule schemes* of $\mathcal{P}(=, 1)$ are:

(11) $\dfrac{\varphi_1 = 1, \ldots, \varphi_n = 1}{\psi = 1}$, whenever $\dfrac{\varphi_1, \ldots, \varphi_n}{\psi}$ is a rule schema of $\mathcal{P}$.

(12) $\dfrac{t_1 = t_2}{t_2 = t_1}, \quad \dfrac{t_1 = t_2,\ t_2 = t_3}{t_1 = t_3}, \quad \dfrac{t_1 = t_1', \ldots, t_n = t_n'}{o_j t_1 \cdots t_n = o_j t_1' \cdots t_n'}$

where $t_1, t_2, \ldots$ are any terms, o_j any n-ary operation symbol.

We also note that:

(13) In the case of the theory $\mathcal{P}(=, 1)$ it is supposed that $P_1, P_2, \ldots$ may be replaced by any terms. In such a way, for example, from axiom-schemes of $\mathcal{P}(=, 1)$ we get the axioms of $\mathcal{P}(=, 1)$.

There is an interesting connection betveen the theories $\mathcal{P}$, $\mathcal{P}(=, 1)$:

(14) An algebra is a model for the calculus $\mathcal{P}$ if and only if M is a model[3] for the theory $\mathcal{P}(=, 1)$.

This immediately follows from the definitions of models for $\mathcal{P}(=, 1)$, and from (10), (12).

Further, we note that the theory $\mathcal{P}(=, 1)$ always has at least one adequate algebra. Namely, these are the so called *free algebras*. One of such algebras Φ can be described in the following way.

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[3] As a matter of fact, its *normal* model, i.e. a model in which $=$ is interpreted as the equality,

Let T be the set of all terms. On the set T define the relation $\sim$ as follows:

(15) $$t_1 \sim t_2 \text{ if and only if } \underset{\mathcal{P}(=,\,1)}{\vdash} t_1 = t_2$$

i.e. $t_1 \sim t_2$ if and only if t_2 can be obtained from t_1 using the axioms of the forms (9), (10) and the rules (11), (12). The relation $\sim$ is an equivalence relation. This follows from the axiom of the form $t=t$, and the rules

$$\frac{t_1 = t_2}{t_2 = t_1}, \qquad \frac{t_1 = t_2,\ t_2 = t_3}{t_1 = t_3}$$

The members of $T/\sim$ i.e. of the quotient set will be denoted by C_t, where $t \in T$. These classes are the only elements of the algebra Φ. Its operations $\bar{o}_j$ are defined as follows:

If $l(o_j) = 0$, then $\bar{o} \overset{\text{def}}{=} C_{oj}$. So $\bar{1} = C_1$.

If $l(o_j) = n$ (>0), then $\bar{o}_j$ is the n-ary operation determined by

(16) $$\bar{o}_j(C_{t_1}, \ldots, C_{t_n}) = C_{o_j t_1 \ldots t_n}.$$

In the definition (16) the result, i.e. $\bar{o}_j(C_{t_1}, \ldots, C_{t_n})$ is defined by $t_1, \ldots, t_n$, i.e. by certain membars of the classes $C_{t_1}, \ldots, C_{t_n}$ respectively. Logical correctness of this definition follows from the rule schema

$$\frac{t_1 = t_1', \ldots, t_n = t_n'}{o_j t_1 \cdots t_n = o_j t_1' \cdots t_n'}$$

The set $T/\sim$ and the operation $\bar{o}_j$ determine the algebra Φ.

The equality (16) can be generalized in the following way.

By

$$t(a_1, \ldots, a_n, o_{j_1}, \ldots, o_{j_m})$$

denote a term, where $a_1, \ldots, a_n$ as well as $o_{j_1}, \ldots, o_{j_m}$ are all elements of the set L i.e. of the set O_1, occurring in the term. Then by induction on m the following equality may easily be proved

(17) $$t(C_{a_1}, \ldots, C_{a_n}, \bar{o}_{j_1}, \ldots, \bar{o}_{j_m}) = C_{t(a_1, \ldots, a_n, o_{j_1}, \ldots, o_{j_m})}$$

We now give a proof that the algebra Φ is an adequate model of the theory $\mathcal{P}(=, 1)$:

The formula

$$t_1(a_1, \ldots, a_n, o_{j_1}, \ldots, o_{j m_1}) = t_2(b_1, \ldots b_{n_2}, o_{k_1}, o_{k_2}, \ldots o_{k m_2})$$

$$(a_1, \ldots, b_1, \ldots \in L;\ o_{j_1}, o_{jk_1}, \ldots \in O_1)$$

is vallid in Φ.

if and only if the equality

$$t_1(C_{x_1}, \ldots, C_{x_{n_1}}, \bar{o}_{j_1}, \ldots, \bar{o}_{j_{m_1}}) = t_2(C_{y_1}, \ldots, C_{y_{n_2}}, \bar{o}_{k_1}, \ldots, \bar{o}_{k_{m_2}})$$

holds in Φ for all $x_1, \ldots, x_{n_1}, y_1, \ldots y_{n_2} \in T$.

if and only if the equality

$$C_{t_1(x_1, \ldots, x_{n_1}, o_{j_1}, \ldots, o_{j_{m_1}})} = C_{t_2(y_1, \ldots, y_{n_2}, o_{k_1}, \ldots, o_{k_{m_2}})}$$

holds in Φ for all $x_1, \ldots, x_{n_1}, y_1, \ldots, y_{n_2} \in T$.

if and only if

$$t_1(x_1, \ldots, x_{n_1}, o_{j_1}, \ldots, o_{j_{m_1}}) \sim t_2(y_1, \ldots, y_{n_2}, o_{k_1}, \ldots, o_{k_{m_2}})$$

for all $x_1, \ldots, x_{n_1}, y_1, \ldots, y_{n_2} \in T$.

if and only if

$$\underset{\mathcal{P}(=,1)}{\vdash} t_1(x_1, \ldots, x_{n_1}, o_{j_1}, \ldots, o_{j_{m_1}}) = t_2(y_1, \ldots, y_{n_2}, o_{k_1}, \ldots, o_{k_{m_2}})$$

for all $x_1, \ldots, x_{n_1}, y_1, \ldots, y_{n_2} \in T$.

if and only if

$$\underset{\mathcal{P}(=,1)}{\vdash} t_1(a_1, \ldots, a_{n_1}, o_{j_1}, \ldots, o_{j_{m_1}}) = t_2(b_1, \ldots, b_{n_2}, o_{k_1}, \ldots, o_{k_{m_2}})$$

The last step in the proof is based on the fact that in any theorem of $\mathcal{P}(=, 1)$, say $\varphi(a_1, a_2, \ldots)$, where $a_1, a_2, \ldots \in L$, the propositional letters $a_1, a_2, \ldots$ may be replaced by any terms (see (13)).

Now we proye our main result.

Theorem. *For a propositional calculus $\mathcal{P}$ there is an adequate algebra if and only if the theory $\mathcal{P}(=, 1)$ is not a creative extension of $\mathcal{P}$, i.e. for every formula f of $\mathcal{P}$ the equivalence*[4] *is true*

$$\underset{\mathcal{P}}{\vdash} f \text{ if and only if } \underset{\mathcal{P}(=,1)}{\vdash} f = 1 \quad (18)$$

[4] The implication

$$\text{If } \underset{\mathcal{P}}{\vdash} f, \text{ then } \underset{\mathcal{P}(=,1)}{\vdash} f = 1$$

is true. Therefore the equivalence (18) may be reduced to the following condition

There is no formula f of $\mathcal{P}$ such that

$$\underset{\mathcal{P}(=,1)}{\vdash} f = 1, \text{ but not } \underset{\mathcal{P}}{\vdash} f$$

Proof. First, suppose that $\mathcal{P}(=, 1)$ is a creative extension of $\mathcal{P}$. Then there is a formula f of $\mathcal{P}$ such that:

$$\vdash_{\mathcal{P}(=,\,1)} f=1 \text{ and not } \vdash_{\mathcal{P}} f$$

Let us also suppose that the calculus $\mathcal{P}$ has an adequate algebra, say M. This algebra is a model of the theory $\mathcal{P}(=, 1)$ as well. Therefore, it follows:

$$f=1 \text{ is valid in } M$$

Hence, we infer that $\vdash_{\mathcal{P}} f$. But from $\vdash_{\mathcal{P}} f$ and not $\vdash_{\mathcal{P}} f$ we conclude that the implication

$\mathcal{P}$ has an adequate model

$\Rightarrow$ $\mathcal{P}(=, 1)$ is not a creative extension of $\mathcal{P}$

is true.

Second, suppose that $\mathcal{P}(=, 1)$ is not a creative extension of. According to the given proof there is an algebra Φ being an adequate algebra of $\mathcal{P}(=1)$. But this algebra is adequate for $\mathcal{P}$ as well, which follows from the proof: Let $f(a_1, \ldots, o_{j_1}, \ldots)$ be any formula of $\mathcal{P}$. Then:

$$f(a_1, \ldots, o_{j_1}, \ldots) \text{ is valid in } \Phi$$

if and only if the equality

$$f(C_{x_1}, \ldots, \bar{o}_{j_1}, \ldots)=C_1$$

holds in Φ for all $x_1, \ldots \in T$

if and only if the equality

$$C_{f(x_1, \ldots, o_{j_1}, \ldots)}=C_1$$

holds in Φ for all $x_1, \ldots \in T$

if and only if

$$\vdash_{\mathcal{P}(=,\,1)} f(x_1, \ldots, o_{j_1}, \ldots)=1$$

hold in Φ for all $x_1, \ldots \in T$

if and only if

$$\vdash_{\mathcal{P}(=.\,1)} f(a, \ldots, o_{j_1}, \ldots)=1$$

if and only if

$$\vdash_{\mathcal{P}} f(a_1, \ldots, o_{j_1}, \ldots)$$