The twistor space of a four-dimensional manifold with a neutral signature

Srdjan Vukmirović

AMS Classification: 81R25, 53C28
email: vsrdjan@matf.bg.ac.yu
Address:
Ohridska 9/4
11080 Zemun
YUGOSLAVIA

Abstract
In this paper we investigate the twistor space \( Z(X) \) of a four-dimensional oriented manifold \( X \) with a metric \( g \) of the neutral signature, using the spinor technique. An almost complex structure \( J \) on \( Z(X) \) is defined and proved that anti-self-duality of \( (X, g) \) is necessary and sufficient for the integrability of \( J \). Some important examples of the twistor spaces are considered in details. We also define a family \( g_\lambda \) of metrics on \( Z(X) \) and prove results analogous to the Riemannian case.

1 Introduction

The basic idea of the Penrose’s twistor programme was to reformulate the problems of the geometry of a real manifold \( X \) into appropriate problems of the complex manifold \( Z(X) \) associated to \( X \). The idea turned out to be very fruitful.

The twistor space of a Riemannian oriented four-dimensional manifold \((X, g)\) was defined in [AHS] as a total space of a bundle \( Z(X) \) whose fiber \( Z_x \) over a point \( x \in X \) is the set of all almost complex structures of the tangent space \( T_x X \) compatible with the metric \( g \) and orientation of \( X \). On such defined manifold \( Z(X) \) the authors defined an almost complex structure which turned to be integrable, i.e. \((Z(X), J)\) is a complex three-dimensional manifold, if and only if \((X, g)\) is self-dual. Further investigations of such twistor spaces are given in [Fri80] and [Fri82] where curvature properties of a family of a metrics \( g_\lambda, \lambda \in \mathbb{R} \) on \( Z(X) \) are studied. The construction of the twistor space \( Z(X) \) is generalized to the case of a quaternionic manifold \( X \) by Salamon in [Sal82]. The notion of the twistor space of a four-dimensional manifold was generalized to the even dimensional (not necessarily Riemannian) manifold in [BO’R]. Special
cases are the four dimensional Riemannian and the quaternionic case. General considerations of the twistor spaces are also given in [Y] and [AG].

The twistor space of a four-dimensional manifold \((X, g)\) with a metric of signature \((3, 1)\) was studied in [Sato]. By reason of similar properties of groups \(SO(4)\) and \(SO(2, 2)\) the twistor space of a neutral four-dimensional manifold \(X\) (with a metric of the signature \((2, 2)\)) enjoys similar properties as Riemannian. As far as we know, it was defined only in [Sal]. The bundle of so called bila-grangian structures over a neutral four-dimensional manifold was investigated in [GJRM].

In this paper we investigate the twistor space of a neutral four-dimensional manifold using the spinor technique from [Fri80] and [Fri82]. This paper has the following structure. Section 2 describes a set of almost complex structures at a point of manifold-the fiber of \(Z(X)\). Section 3 introduces additional notation. Section 4 gives a relation between the almost complex structures on a manifold \(X\) and the sections of its spinor bundle. Section 5 contains a main theorem concerning integrability of certain almost complex structure \(J\) on \(Z(X)\). Section 6 contains an example of the twistor space of the neutral sphere and neutral complex projective plane.

## 2 Almost complex structures

In case of a real four-dimensional vector space \( \mathbb{R}^{0,4} = (\mathbb{R}^{4}, g) \) with a positive definite scalar product \( g \) of the signature \((0, 4)\) we define an almost complex structure \( J : \mathbb{R}^{4} \to \mathbb{R}^{4} \) as a linear map with the following properties:

1. \( J^2 = -I_d \)
2. \( J \) is an isometry i.e. \( g(Jx, Jy) = J(x, y) \),
3. \( J \) preserves the fixed orientation of the vector space \( \mathbb{R}^{4} \).

Set \( \mathcal{J} \) of all such complex structures is sphere \( S^2 \). It naturally carries the complex structure of the complex projective space \( S^2 = \mathbb{CP}^1 = SU(2)/S^1 \). Because of the transitive action of the isometry group \( SO(4) \) on it, \( \mathcal{J} \) can be seen as a homogenous space \( \mathcal{J} = SO(4)/H \).

It is easy to see that in case of vector space \( \mathbb{R}^{1,3} \) with a metric \( g \) of the signature \((1, 3)\), i.e. \((-+++)\) it is not possible to define an almost complex structure \( J \) with the properties 1)-3), or similar.

The case of the real vector space \( \mathbb{R}^{2,2} \) with a metric \( g \) of neutral signature \((2, 2) = (-+++)\) is similar to the definite case. In this case we have notion of space and time orientation of the vector space \( \mathbb{R}^{2,2} \). Isometries with a positive determinant can preserve or reverse both of them at the same time. Hence we have to modify requirement 3) into the following condition:

3‘) \( J \) preserves the fixed orientation of \( \mathbb{R}^{2,2} \) and its space and time orientation.

For the chosen spacelike vector \( e_1 \), \( J(e_1) \) must belong to its pseudoortogonal complement \( e_1^\perp = \mathbb{R}^{2,1} \), or more precisely to one sheet (in order to preserve space orientation) of the pseudosphere of the unit spacelike vectors. We see that

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it is a submanifold of $\mathbb{R}^{2,1}$ with a negative definite induced metric. Preserving time orientation completely determines the almost complex structure $J$ so we have

$$J = \{(x_1, x_2, x_3) \in \mathbb{R}^{2,1} \mid - x_1^2 - x_2^2 + x_3^2 = 1, \ x_3 > 0 \}.$$ 

We have choose $e_1$ to be spacelike since the almost complex structures can be described in terms of the anti-self-dual two-forms which form a vector space of the signature $(2,1)$, as we will see later. 

As in the definite case we can represent $J$ as a homogenous space $SO^+(2,2)/H$ of the group $SO^+(2,2)$ of isometries of $\mathbb{R}^{2,2}$ with a positive determinant preserving both time and space orientation, over its subgroup $H$.

Since $J$ will be the fiber of our twistor bundle, which we want to be a complex manifold, we will show that it is a homogenous space of the group $SU(1,1)$ and investigate it in more details, emphasizing its complex structure. Denote by $I_{p,q}$ the diagonal matrix

$$I_{p,q} := \text{diag}\{-1, \ldots, -1, 1, \ldots, 1\}.$$ 

Pseudounitary group $U(p,q)$ is group of the complex linear transformations preserving the hermitian scalar product given by the matrix $I_{p,q}$

$$U(p,q) := \{ A \in M(p+q, \mathbb{C}) \mid A^\dagger I_{p,q} A = I_{p,q} \},$$

and special pseudounitary group is its subgroup

$$SU(p,q) := \{ A \in U(p,q) \mid \det A = 1 \}.$$ 

**Theorem 2.1** Space $J$ of all almost complex structures satisfying 1), 2) and 3') is the homogenous space $SU(1,1)/U(1)$ on which $SU(1,1)$ acts by the conjugation.

**Proof:**

Because of the properties 1), 2) and 3') the matrix $J$ of any almost complex structure has to be of the form:

$$J = \begin{pmatrix} 0 & \mu_1 & \mu_2 & \mu_3 \\ \mu_1 & 0 & -\mu_3 & \mu_2 \\ -\mu_3 & \mu_2 & 0 & \mu_1 \\ \mu_1 & \mu_3 & -\mu_1 & 0 \end{pmatrix}, \ \mu_1^2 - \mu_2^2 - \mu_3^2 = 1, \ \mu_1 > 0,$$

or in the complex notation

$$J = \begin{pmatrix} i\mu_1 & \bar{\mu} \\ \mu & i\mu_1 \end{pmatrix}, \ \mu = \mu_2 + i\mu_3 = |\mu|e^{im}.$$ 

Any complex structure $J$ can be obtained using the conjugation $J = A^{-1}J_0A$ by the matrix $A \in SU(1,1)$, from the complex structure $J_0$

$$J_0 := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$
In fact, the matrix $A$ is given by:

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} -ie^{i(m-\varphi)}\sqrt{\mu_4 + 1} & e^{-i\varphi}\sqrt{\mu_4 - 1} \\ ie^{i(m-\varphi)}\sqrt{\mu_4 - 1} & e^{i\varphi}\sqrt{\mu_4 + 1} \end{pmatrix}, \quad \varphi, \Psi \in \mathbb{R}, \quad \frac{\pi}{2} + \varphi + \Psi = m. $$

The stability subgroup of $J_0$ is $U(1)$, so we finally have $J = SU(1, 1)/U(1)$.

We see that space of the almost complex structures $\mathcal{J}$ have the same homogenous representation as the set of all positive definite complex lines in the neutral complex vector space $\mathbb{C}^{1,1}$.

Now, we are going to investigate the homogenous space $SU(1, 1)/U(1)$ on a Lie algebra level. Its easy to check that

$$su(1, 1) = \left\{ \begin{pmatrix} it & z \\ \bar{z} & -it \end{pmatrix} \mid t \in \mathbb{R}, \quad z \in \mathbb{C} \right\}. $$

Lie algebra $s^1$ of stationary group $S^1$ can be seen like a subalgebra of $su(1, 1)$ in the following way

$$s^1 = \left\{ \begin{pmatrix} it & 0 \\ 0 & -it \end{pmatrix} \mid t \in \mathbb{R} \right\}. $$

We obtain the reductive decomposition $su(1, 1) = n + s^1$ of Lie algebra $su(1, 1)$ where subspace $n$ represent the tangent space in a point of a manifold

$$n = \left\{ n(z) := \begin{pmatrix} 0 & z \\ \bar{z} & 0 \end{pmatrix} \mid z \in \mathbb{C} \right\}. $$

We naturally define the almost complex structure $j : n \rightarrow n$ by

$$j(n(z)) := n(iz). \quad (1) $$

The negative definite scalar product induced on $m$ is given by

$$< n(z), n(w) > := -\frac{1}{2} \text{tr}(n(z)n(w)). \quad (2) $$

3 (Anti-)self-dual manifolds in signature $(2, 2)$

Let $X^{2,2} = (X, g)$ be an oriented pseudo-Riemannian manifold of the signature $(2, 2)$, i.e. $(-+ + +)$. Locally, there exist an oriented pseudoorthonormal frame $e_1, e_2, e_3, e_4$,

$$e_6 := g(e_6, e_6), \quad e_1 = -1 = e_2, \quad e_3 = 1 = e_4 $$

such that $e_1, e_2$ are time oriented and $e_3, e_4$ are space oriented. Let $e^1, e^2, e^3, e^4$ denotes its dual coframe such that $\mathcal{O} = e^1 \wedge e^2 \wedge e^3 \wedge e^4$ gives an orientation on $X^{2,2}$.  

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For any forms $\nu, \eta$ from space $\Lambda^i := \Lambda^i_x X^{2,2}$ of $i$-forms at a point $x \in X^{2,2}$ we can define their scalar product in the following way

$$(\nu, \eta)_x := \sum_{1 \leq j_1, \ldots, j_i \leq 4} \epsilon_{j_1} \cdots \epsilon_{j_i} \nu(e_{j_1}, \ldots, e_{j_i}) \eta(e_{j_1}, \ldots, e_{j_i})_x.$$ 

We will be especially interested in the space of two forms

$\Lambda^2 = \langle e^{12}, e^{13}, e^{14}, e^{23}, e^{24}, e^{34} \rangle$,

where we use notation $e^{ij} := e^i \wedge e^j$, $1 \leq i < j \leq 4$. It is easy to calculate their scalar products

$$(e^{ij}, e^{kl}) = \begin{cases} 
\epsilon_i \epsilon_j & \text{if } (i, j) = (k, l) \\
0 & \text{otherwise.}
\end{cases}$$

Hence, $\Lambda^2$ is a six dimensional vector space of the signature $(4,2)$. We recall a definition of the Hodge star operator

$\ast : \Lambda^i \to \Lambda^{4-i}$, $\nu \wedge \ast \eta := (\nu, \eta) \mathcal{O}$.

One can check that $\ast$ is a self-adjoint operator with respect to the scalar product $(\cdot, \cdot)$ satisfying $\ast \circ \ast = Id$ on two forms. This involutivity is true in the Riemannian case unlike in case of the signature $(3,1)$ where Hodge operator satisfies the relation $\ast \circ \ast = -Id$.

Because of the involutivity one can decompose space $\Lambda^2$ of two forms into the direct sum of $+1$-eigenspace $\Lambda^2_+$ and $-1$-eigenspace $\Lambda^2_-$, i.e.

$$\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-.$$ 

$\Lambda^2_+$ (resp. $\Lambda^2_-$) is so called space of self-dual (resp. anti-self-dual) forms. Here are their bases together with the notation which we will use

$$\Lambda^2_+ = \langle e^{12} + e^{14}, e^{13}, e^{24}, e^{23} - e^{14} \rangle,$$

$$\Lambda^2_- = \langle e^{12} - e^{14}, e^{13}, -e^{24}, e^{23} + e^{14} \rangle.$$ 

One can check that these are pseudoortonormal basises and that both $\Lambda^2_+$ and $\Lambda^2_-$ are vector spaces of the signature $(2,1)$. If we denote by $\nabla$ the Levi-Civita connection of metric $g$ then the curvature tensor $R$ is defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

with the components $R_{ijkl} := g(R(e_i, e_j)e_k, e_l)$.

We define the Ricci tensor

$$R_{ij} := \sum_k \epsilon_k R_{ikkj}$$

which turns out to be self-adjoint with respect to $g$ and the scalar curvature $\tau$ as the trace of the Ricci tensor

$$\tau := \sum_i \epsilon_i R_{ii} = \sum_{i,j} \epsilon_i \epsilon_j R_{jij}.$$ 

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We say that a manifold $X^{2,2}$ is an *Einstein space* if the Ricci tensor is proportional to the metric tensor $g$ (in that case the proportional factor is constant over $X^{2,2}$).

The curvature tensor can be seen like a bundle morphism $R : \Lambda^2 \to \Lambda^2$ given by

$$R(e^{ij}) = \sum_{k<l} \epsilon_k \epsilon_l R_{ijkl} e^{kl}.$$ 

One can show that $R$ is self-adjoint with respect to the metric in $\Lambda^2 X$. With respect to the decomposition $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$ it has the following block-matrix form

$$R = \begin{pmatrix} A & B \\ D & C \end{pmatrix},$$

where $A : \Lambda^2_+ \to \Lambda^2_+$ and $C : \Lambda^2_- \to \Lambda^2_-$ are self-adjoint endomorphisms and $D = B^* : \Lambda^2_+ \to \Lambda^2_-$ is map adjoint to $B$. Following [Bese] we have further decomposition

$$R = \begin{pmatrix} 0 & B \\ D & 0 \end{pmatrix} + \begin{pmatrix} A + \frac{\tau}{12} & 0 \\ 0 & C + \frac{\tau}{12} \end{pmatrix} - \frac{\tau}{12} I_6$$

where

$$W = W_+ \oplus W_- = (A + \frac{\tau}{12}) \oplus (C + \frac{\tau}{12})$$

is Weil tensor.

It is self-adjoint, traceless, commutes with Hodge star operator, have the same symmetry properties like the curvature tensor and is invariant to a conformal changes of metric.

We say that a manifold $X^{2,2}$ is *self-dual* (resp. *anti-self-dual*) if $W_- = 0$ (resp. $W_+ = 0$). One can show the following proposition

**Lemma 3.1** Manifold $X^{2,2}$ is anti-self-dual if and only if following relations holds

$$\begin{align*}
\frac{\tau}{6} &= R_{2112} + R_{4334} - 2(R_{1324} - R_{1423}) \\
\frac{\tau}{6} &= -R_{3113} - R_{4224} + 2R_{1324} \\
\frac{\tau}{6} &= -R_{3223} - R_{4114} - 2R_{1423} \\
0 &= -R_{2113} + R_{1334} + R_{1224} - R_{2443} \\
0 &= R_{1223} + R_{3334} + R_{2114} + R_{1443} \\
0 &= -R_{1332} - R_{3224} + R_{3114} + R_{1442}.
\end{align*}$$

Notice that previous six equations are dependent. One can use only five of them, but we found that six are more convenient.
4 The description of almost complex structures in terms of spinors

In this section we are going to prove one-to-one correspondence between almost complex structures on a manifold \(X\) and sections of bundle of the projective spinors.

4.1 The spinor representation of a group \(Spin(2,2)\)

Let \(C_{2,2}\) denote the Clifford algebra of the real vector space \(\mathbb{R}^{2,2}\) with a metric of the signature \((2,2)\). It is by definition an algebra with a unity generated by pseudoortonormal vectors \(e_1, e_2, e_3, e_4\) satisfying the algebra relations

\[
e_i \cdot e_j + e_j \cdot e_i = -2\epsilon_{ij}\delta_{ij}.
\]

In our four-dimensional case its complexification \(\mathbb{C}C_{2,2}\) is isomorphic (like a vector space) to the complex matrix algebra \(M(4,\mathbb{C}) = M(2,\mathbb{C}) \otimes M(2,\mathbb{C})\). One can check that an algebra isomorphism \(H : \mathbb{C}C_{2,2} \to M(4,\mathbb{C})\) is given by the mapping

\[
H(e_1) := iE \otimes U, \quad H(e_2) := iE \otimes V, \quad H(e_3) := U \otimes T, \quad H(e_4) := V \otimes T.
\]

Vectors \(u(\epsilon_1, \epsilon_2) = u_{\epsilon_1} \otimes u_{\epsilon_2}, \quad \epsilon_1, \epsilon_2 \in \{1, -1\}, \quad u_1 = \begin{pmatrix} 1 \\ 0 \\ -i \\ 1 \end{pmatrix}, \quad u_{-1} = \begin{pmatrix} 1 \\ -i \\ 0 \\ 1 \end{pmatrix}\) form a basis of the vector space \(\mathbb{C}^4\).

The action of \(H(\mathbb{C}C_{2,2})\) on \(\mathbb{C}^4 = \mathbb{C}^2 \otimes \mathbb{C}^2\) is given by

\[(A \otimes B)(u \otimes v) := Au \otimes Bv, \quad A \otimes B \in M(2,\mathbb{C}) \otimes M(2,\mathbb{C}), \quad u \otimes v \in \mathbb{C}^2 \otimes \mathbb{C}^2.\]

Vectors

\[u(\epsilon_1, \epsilon_2) = u_{\epsilon_1} \otimes u_{\epsilon_2}, \quad \epsilon_1, \epsilon_2 \in \{1, -1\}, \quad u_1 = \begin{pmatrix} 1 \\ 0 \\ -i \\ 1 \end{pmatrix}, \quad u_{-1} = \begin{pmatrix} 1 \\ -i \\ 0 \\ 1 \end{pmatrix}\]

form a basis of the vector space \(\mathbb{C}^4\).

The action of \(x \in \mathbb{C}C_{2,2}\) on \(v \in \mathbb{C}^4\) we will denote by

\[x \cdot v := H(x)(v).\]

Now, we decompose \(\Delta = \Delta^+ \oplus \Delta^-\) where

\[\Delta^+ = \langle u(1, 1), u(-1, -1) \rangle\]

\[\Delta^- = \langle u(-1, 1), u(1, -1) \rangle.\]

Following [Baum] we introduce a Hermitian scalar product in \(\Delta\) such that basis \(u(\epsilon_1, \epsilon_2)\) is pseudoortonormal and

\[\|u(\epsilon_1, \epsilon_2)\|^2 := \epsilon_2.\]
In that way both $\Delta^+$ and $\Delta^-$ become the complex vector spaces of the real signature $(2,2)$. We proceed with a standard definition of the spin group

$$\text{Spin}(2,2) := \{ x \cdot y \mid x, y \in \mathbb{R}^{2,2}, \ ||x||^2 = \pm 1 = ||y||^2 \} \subset \mathbb{C}^{2,2}. $$

The Lie algebra of group $\text{Spin}(2,2)$ is a linear hull

$$\text{spin}(2,2) = \langle \{ e_i \cdot e_j \mid 1 \leq i < j \leq 4 \} \rangle \subset \mathbb{C}^{2,2}. $$

We will denote the restriction of the representation $\mathcal{H}$ on $\text{Spin}(2,2)$ by the same letter. Since $\mathcal{H}$ is linear, its differential $\mathcal{H}^* : \text{spin}(2,2) \to \mathfrak{gl}(\Delta)$ satisfies $\mathcal{H}^* = \mathcal{H}$. One can prove that $\mathcal{H}$ is an exact unitary representation of the group $\text{Spin}(2,2)$, i.e. $\mathcal{H}(\text{Spin}(2,2)) \subset SU(2,2)$. Moreover, noticing that $\mathcal{H}$ preserves the decomposition $\Delta = \Delta^+ \oplus \Delta^-$ we can decompose the representation $\mathcal{H}$ into the direct sum of two representations

$$\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^- : \text{Spin}(2,2) \to SU(\Delta^+) \oplus SU(\Delta^-) = SU(1,1) \oplus SU(1,1).$$

Mapping $\rho : \text{Spin}(2,2) \to SO(2,2)$ given by

$$\rho(y)x := y \cdot x \cdot y^{-1}, \ y \in \text{Spin}(2,2), \ x \in \mathbb{R}^{2,2}$$

is the double covering homomorphism. This facts become more clear after showing that $\rho(y)$ represent a reflection of $\mathbb{R}^{2,2}$ with respect to the hyperplane with pseudonormal vector $y$.

Lie algebra of the Lie group $SO(2,2)$ is

$$\text{so}(2,2) = \langle \{ E_{ij} := \varepsilon_i M_{ji} - \varepsilon_j M_{ij} \mid 1 \leq i < j \leq 4 \} \rangle, \quad (3)$$

where $M_{ij}$ is a matrix whose all elements are zero except the element in $i$-th row and $j$-th column which is equal one.

The differential $\rho_* : \text{spin}(2,2) \to \text{so}(2,2)$ is given by

$$\rho_*(e_i \cdot e_j) = 2E_{ij},$$

since $\rho(y)$ represent the reflection with respect to the hyperplane with normal vector $y$.

Now we have a composition of two mappings

$$\text{SO}(2,2) \xrightarrow{\rho} \text{Spin}(2,2) \xrightarrow{\mathcal{H}} SU(1,1).$$

by which the group $\text{SO}(2,2)$ acts (see section 2) on the set of all almost complex structures $\mathcal{J} = SU(1,1)/S^1$. It’s not difficult to calculate the Lie algebra $h^\pm$ of the isotropy group $H^\pm \subset \text{SO}(2,2)$, of that action

$$h^\pm = (\mathcal{H}^\pm \circ \rho^{-1})_*^{-1}(s^1).$$
is
\[ h^\pm = \{ \sum_{i<j} a_{ij} E_{ij} \mid a_{14} \pm a_{23} = 0, \ a_{13} \mp a_{24} = 0 \}. \]

If we denote by \( n^\pm \) its complement in \( so(2, 2) \) we have
\[ so(2, 2) = h^\pm \oplus n^\pm. \]  

We can carry the natural complex structure \( J \) given by the formula (1) from \( P(\Delta^\pm) \) to the complex structure \( J^\pm \) on \( n^\pm \):
\[
\begin{align*}
  j^\pm(E_{14} \pm E_{23}) &= E_{13} \mp E_{24} \\
  j^\pm(E_{13} \mp E_{24}) &= -(E_{14} \pm E_{23}).
\end{align*}
\]

4.2 A description of almost complex structures as sections of \( P^- \)

Denote by \((Q, \pi, X^{2,2}, SO(2, 2))\) the principal \( SO(2, 2) \) bundle of the manifold \( X^{2,2} \). Recall that a spin structure \((\tilde{Q}, \pi, X^{2,2}, Spin(2, 2))\) is a principal \( Spin(2, 2) \) bundle together with a principal bundle homomorphism \( H : \tilde{Q} \to Q \) such that
\[
H(\tilde{q} \cdot x) = H(\tilde{q}) \cdot \rho(x), \ x \in Spin(2, 2), \ \tilde{q} \in \tilde{Q},
\]
where multiplication \( \cdot \) denotes an action of the structure group on the bundle. Not all manifolds admit a spin structure, but exactly those with vanishing second Stiefel-Whitney class.

In the previous section we said that the group \( SO(2, 2) \) acts on the projectivization \( P(\Delta^\pm) \) of the spinor bundle by the conjugation. Hence, we can consider the fiber bundle
\[ P^\pm = Q \times_{SO(2, 2)} P(\Delta^\pm). \]

Recall that by a definition bundle \( P^\pm \) have the same transition functions as \( Q \) and that its fiber is set of equivalence classes \( \{ [g, v] \mid g \in Q, \ v \in P(\Delta^\pm) \} \) where the relation of the equivalence is given by \( (g, v) \sim (gh^{-1}, hv), \ h \in SO(2, 2). \)

Locally, in some region \( U \subset X^{2,2} \) diffeomorphic to \( \mathbb{R}^4 \), we can always choose some spin structure \((\tilde{Q}, \pi, U, Spin(2, 2))\) which determines the spinor bundles
\[ S^\pm_U := \tilde{Q} \times_{Spin(2, 2)} \Delta^\pm, \]
which are vector bundles with the canonical projection onto corresponding projectivizations
\[ h : (S^\pm \setminus 0)_U \to P^\pm_U. \]

For every section \( \Phi^\pm \) of the projective spinor bundle \( P^\pm \), i.e. the mapping
\[ \Phi^\pm : X^{2,2} \supset U \to P^\pm, \]
choose a section \( \tilde{\Phi}^\pm \) of the spinor bundle \( (S^\pm \setminus 0)_U \) such that \( h(\tilde{\Phi}^\pm) = \Phi^\pm \).
After choosing such sections, using Clifford multiplication, we get the following isomorphism
\[(TX^{2,2})_U \ni t \mapsto t \cdot \tilde{\Phi}^\pm \in (S^\pm \setminus 0)_U\]
between vector fields and spinors.

By reason of that isomorphism, for every section $\Phi^\pm$ we can carry the natural almost complex structure from $S^\pm$ (which fiber is a complex vector space) to obtain an almost complex structure $J^{\Phi^\pm} : T(X^{2,2})_U \to T(X^{2,2})_U$ of the tangent space. Namely, we define
\[J^{\Phi^\pm}(t) \cdot \tilde{\Phi}^\pm := i(t \cdot \tilde{\Phi}^\pm). \tag{6}\]
This definition of $J^{\Phi^\pm}$ does not depend of the choice of $\tilde{\Phi}^\pm$, so it is well defined not only over $U$, but globally over $X^{2,2}$.

**Theorem 4.1** Let $\Phi^\pm$ be a section in the projective spinor bundle $P^\pm$. Then

1. $J^{\Phi^\pm}$ is an almost complex structure (with properties 1) and 2) from the section 2) which preserves space and time orientation.

2. $J^{\Phi^-}$ induces given orientation on $X^{2,2}$ while $J^{\Phi^+}$ induces the opposite orientation.

3. Set of the all almost complex structures inducing given (resp. opposite) orientation is parameterized by the sections of $\Phi^-$ (resp. $\Phi^+$).

## 5 An almost complex structure on $P^-$ and its integrability

We have described the projective spinor bundle $P^- = Q \times_{SO(2,2)} P(\Delta^-)$ of a manifold $X^{2,2}$ and have related its sections to the almost complex structures on $X^{2,2}$. Now we are going to define a natural almost complex structure $J$ on it and to show the main result - that $J$ is integrable (i.e. $(P^-, J)$ is a complex manifold) if and only if $X^{2,2}$ is anti-self-dual.

The Levi-Civita connection on the principal $SO(2,2)$ bundle $Q$ over $X^{2,2}$ induces decomposition
\[TP^- = T_h P^- \oplus T_v P^- \tag{7}\]
of the tangent space of the bundle $P^-$ into its vertical and horizontal component.

Every vector field $X$ on $P$ can be uniquely decomposed as a sum of its horizontal and vertical part
\[X = X_h + X_v.\]
We will denote by $\sigma(X) \in T_h P^-$ the fundamental vector field corresponding to the vector $X$ from the tangent space $n$ of $SU(1,1)/S^1$.

We are going to define an almost complex structure $J$ on $TP^-$ first on the vertical and then on the horizontal part.
Since the fiber $SU(1,1)/S^1$ of the bundle $P^-$ carries the natural almost complex structure given by (1) which is invariant under the $SO(2,2)$ action we can choose it to be a vertical component of the almost complex structure $J$

$$J(\sigma(X)) := \sigma(j(X)).$$

Recall that sections $\Phi^- : X^{2,2} \to P^-$ of $P^-$ parameterize almost complex structures on $X^{2,2}$. For a horizontal vector $v$ in a point $p \in P^-$ we define $J(v)$ using any section $\Phi^-$ satisfying $\Phi^-(\pi(p)) = p$, as a unique horizontal vector such that

$$\pi_*J(v) = J^{\Phi^-}(\pi(p))\left(\pi_*(t)\right),$$

independently of the section $\Phi^-$. Roughly speaking, we have defined an almost complex structure in a point $p \in P^-$, as those structure, parameterized by the $p$ in the horizontal part and canonically in the vertical part of the tangent space $T_pP^-$.  

**Theorem 5.1** $J$ is integrable (ie. $P^-$ is a complex manifold) if and only if the manifold $X^{2,2}$ is anti-self-dual.

Since the notion of anti-self-duality and self-duality can be interchanged simply by changing the orientation of the manifold $X^{2,2}$ theorem holds for a self-dual manifold with the opposite orientation.

### 6 Examples and remarks

Here are given two basic examples- the twistor space of the neutral complex projective plane and the neutral four-dimensional sphere. They are anti-self-dual and hence their twistor spaces are complex manifolds. A natural family $g_\lambda, \lambda \in \mathbb{R}$ of metrics on the twistor space $P^-$ is defined and stated its curvature properties similar to those in Riemannian case given in [Fri82].

The complex projective plane (set of negative definite lines in the complex space $\mathbb{C}^{2,1}$) is a homogenous space

$$\mathbb{C}P^{1,1} = SU(2,1)/U(1,1).$$

From the Theorem 2.1 we know that the group $SU(1,1)$ acts transitively by the conjugation on the set $\mathcal{J}$ of almost complex structures of the space $\mathbb{R}^{2,2}$, so it does a bigger group $U(1,1) = SU(1,1) \times U(1)$. The group $SU(2,1)$ acts transitively on the twistor space $P^- (\mathbb{C}P^{1,1})$. An action of the group $SU(2,1)$ on the twistor space

$$P^- (\mathbb{C}P^{1,1}) = \{(x, J_x) \mid x \in \mathbb{C}P^{1,1}, J_x \in \mathcal{J}(T_x \mathbb{C}P^{1,1}) = \mathcal{J}\},$$

defined by

$$A \cdot (x, J_x) = (Ax, A^{-1}J_x A), \ A \in SU(2,1)$$

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is transitive. The isotropy group of this action is $U(1) \times U(1)$ and hence we have

$$P^-(\mathbb{C}P^{1,1}) = SU(2, 1)/(U(1) \times U(1)).$$

We see that the twistor space of the neutral complex projective plane is a non-compact flag manifold.

The neutral sphere $S^{2,2}$ can be represented like a hypersurface

$$S^{2,2} = \{(x_1, ..., x_5) \in \mathbb{R}^{2,3} \mid 1 = -x_1^2 - x_2^2 + x_3^2 + x_4^2 + x_5^2\}$$

of $\mathbb{R}^{2,3}$ (diffeomorphic to $\mathbb{R}^2 \times S^2$).

Because of the transitive isometric action of $SO(2, 3)$ it can be seen as a homogeneous space

$$S^{2,2} = SO^+(2, 3)/SO^+(2, 2).$$

Here $SO^+(p, q)$ denotes the connected component of unity of the group $SO(p, q)$, that is, the group of those isometries of $\mathbb{R}^{p,q}$ which preserves both time and space orientation. By definition of the almost complex structure the group $SO^+(2, 2)$ acts transitively by the conjugation on the set $\mathcal{J}$ of almost complex structures. Like in the previous case the twistor space is a homogenous space

$$P^-(S^{2,2}) = SO^+(2, 3)/H$$

where the isotropy subgroup $H$ of an element $(e_5 := (0,0,0,0,1), J_0) \in P^-(S^{2,2})$, is given by

$$H = \{ A \in SO^+(2, 2) \mid AJ_0 = J_0A \}, \quad J_0 := \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Now, we are going to define a family of metrics on $P^-(S^{2,2})$. We can decompose algebra $so(2, 3)$ into the direct sum

$$so(2, 3) = m \oplus n \oplus h,$$

where $m$ is identified with the tangent space on the neutral sphere in the point $[e_5]$ and $n$ represents the tangent space on the fiber in that point. Denote by $B$ Killing form on the Lie group $SO(2, 3)$

$$B(X, Y) := -\frac{1}{2}tr(XY), \quad X, Y \in so(2, 3).$$

We define a scalar product $g_\lambda, \lambda \in \mathbb{R}$ on $m \oplus n$ by

$$g_\lambda(X, Y) := B(X, Y)|_m, \quad X, Y \in m,$$

$$g_\lambda(X, Y) := \lambda B(X, Y)|_n, \quad X, Y \in n.$$

It defines $SO^+(2, 3)$ invariant scalar product $g_\lambda$ on $P^-(S^{2,2})$. 

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The complex structure $J : m \oplus n \rightarrow m \oplus n$ is given on $m$ by the matrix $J_0$ in the basis $X_1, X_2, X_3, X_4$, and up to sign canonically by $J(X_5) = X_6$ on $n$. It is integrable and becomes an invariant complex structure on $P^-(S^{2,2})$. If we now calculate the curvature tensor, we will see that the metric $g_\lambda$ is Einstein for $\lambda = 1$ or $\lambda = 2$. The manifold $(P^-(S^{2,2}), g_\lambda, J)$ is a Kähler-Einstein only for $\lambda = 2$. One can check that an almost complex structure $J'$ defined by $J'(X_5) = -X_6$ on $n$ and as $J$ on $m$, is not integrable.

The similar is true in a general situation. In the Riemannian case it was proved in [Fri82] and [Fri85]. Since the calculation is very similar in the neutral case, only a statement will be formulated without a proof.

We will define a family $g_\lambda, \lambda \neq 0$ of metrics on $P^-$ first on the vertical, then on the horizontal tangent subspace according to the decomposition (7). A fiber of the twistor bundle carries the natural definite metric

$$g_\lambda(\sigma(X), \sigma(Y)) := \lambda < X, Y >, \ X, Y \in n$$

where $< \cdot, \cdot >$ is Fubini-Stud metric given by 2. For horizontal vector fields $X, Y$ on $P^-$ we define

$$g_\lambda(X, Y) := \pi^*g(X, Y), \ X, Y \in T_hP^-,$$

where $g$ is the metric on $X^{2,2}$. One can show that the following theorem is true

**Theorem 6.1** $(P^-(X^{2,2}), J, g_\lambda)$, $\lambda \neq 0$ is an Einstein manifold if and only if $\lambda \tau = 48$ or $\lambda \tau = 30$, where $\tau$ is nonzero scalar curvature of $X^{2,2}$. It is a Kähler manifold if and only if $\lambda \tau = 48$.

An important examples of self-dual manifolds of the neutral signature are Osserman manifolds. In [ABBR] it was shown that $(X, g)$ is Osserman if and only if $(X, g)$ is (anti-)self-dual and Einstein.

### References


[Sal] Salamon S., An indefinite twistor space, Lecture notes, 5 pages
