Equivariant Methods

Siniša Vrećica

University of Belgrade

Computational Applications of Algebraic Topology

Topological methods in combinatorics, computational geometry, and the study of algorithms

MSRI, October, 2006.
$G$-spaces and equivariant maps

- $X$ is a $G$-space if $G < \text{Homeo } X$, i.e. if there is a family of homeomorphisms of the space $X$ denoted by the elements of the group $G$ which compose as the elements of group $G$ multiply. We say that $G$ acts on $X$. 
$G$-spaces and equivariant maps

- $X$ is a $G$-space if $G \lt \text{Homeo } X$, i.e. if there is a family of homeomorphisms of the space $X$ denoted by the elements of the group $G$ which compose as the elements of group $G$ multiply. We say that $G$ acts on $X$.

- $f : X \rightarrow Y$ is an equivariant map (or a $G$-map) if 
  \[(\forall g \in G)(\forall x \in X) f(g \cdot x) = g \cdot f(x),\]
  i.e. if $f$ commutes with the action of the elements of the group $G$ (we say that $f$ preserves $G$-symmetry).
Borsuk-Ulam theorem

\textbf{Theorem:} For every continuous map 
\( f : S^n \rightarrow \mathbb{R}^n \) there is \( x \in S^n \) such that 
\( f(-x) = f(x) \).
Borsuk-Ulam theorem

**Theorem:** For every continuous map $f : S^n \to \mathbb{R}^n$ there is $x \in S^n$ such that $f(-x) = f(x)$.

**Theorem:** For every $\mathbb{Z}/2$-equivariant map $f : S^n \to \mathbb{R}^n$ there is $x \in S^n$ such that $f(x) = 0$. 
Borsuk-Ulam theorem

- **Theorem:** For every continuous map $f : S^n \rightarrow \mathbb{R}^n$ there is $x \in S^n$ such that $f(-x) = f(x)$.

- **Theorem:** For every $\mathbb{Z}/2$-equivariant map $f : S^n \rightarrow \mathbb{R}^n$ there is $x \in S^n$ such that $f(x) = 0$.

- **Theorem:** There is no $\mathbb{Z}/2$-equivariant map $f : S^n \rightarrow S^{n-1}$. 
Borsuk-Ulam theorem, continued

- one of the most often applied topological results in geometry and combinatorics
Borsuk-Ulam theorem, continued

- one of the most often applied topological results in geometry and combinatorics

- **HAM-SANDWICH THEOREM**: *For every $n$ measurable sets in $\mathbb{R}^n$, there is a hyperplane dividing each of them in two parts of the same measure.*
Borsuk-Ulam theorem, continued

- one of the most often applied topological results in geometry and combinatorics

- **Ham-Sandwich Theorem**: For every $n$ measurable sets in $\mathbb{R}^n$, there is a hyperplane dividing each of them in two parts of the same measure.

Some other applications

- L. Lovász established the Kneser conjecture from Combinatorics in 1978 showing:

**THEOREM:** However we divide the family of all \( n \)-subsets of the set of \( 2n + k \) elements in the \( k + 1 \) subfamilies, there is a pair of disjoint \( n \)-subsets in some of the subfamilies.
Some other applications

- L. Lovász established the Kneser conjecture from Combinatorics in 1978 showing:

**Theorem:** However we divide the family of all \( n \)-subsets of the set of \( 2n + k \) elements in the \( k + 1 \) subfamilies, there is a pair of disjoint \( n \)-subsets in some of the subfamilies.

- The proof is based on the extremely clever application of the Borsuk-Ulam theorem.
Some other applications, continued

I. Bárány, S. Shlosman and A. Szücs extended in 1981 the Tverberg theorem from affine to the continuous case proving:
I. Bárány, S. Shlosman and A. Szücs extended in 1981 the Tverberg theorem from affine to the continuous case proving:

**Theorem:** Let $p$ be a prime integer and $N = (p - 1)(n + 1)$. For every continuous map $f : \Delta^N \rightarrow \mathbb{R}^n$ there are $p$ disjoint faces of $\Delta^N$ whose $f$-images intersect.
Some other applications, continued

- I. Bárány, S. Shlosman and A. Szücs extended in 1981 the Tverberg theorem from affine to the continuous case proving:

  **THEOREM:** Let \( p \) be a prime integer and \( N = (p - 1)(n + 1) \). For every continuous map \( f : \Delta^N \rightarrow \mathbb{R}^n \) there are \( p \) disjoint faces of \( \Delta^N \) whose \( f \)-images intersect.

- M. Ozaydin (later K. Sarkaria, A. Volovikov) generalized the result for prime-powers, and showed that the method does not work in other cases.
Some other applications, continued

Some other applications, continued


- D. Kozlov, in his lecture, said more about the recent resolution of the Lovász conjecture by E. Babson and himself.
Scheme

$X = \text{configuration space} = \text{space of all candidates}$
Scheme

- $X = \text{configuration space} = \text{space of all candidates}$
- $(Y, A) = \text{test space with subspace}$
**Scheme**

- $X = \text{configuration space} = \text{space of all candidates}$
- $(Y, A) = \text{test space with subspace}$
- The symmetry of the problem induces an action of some group $G$ both on $X$ and $(Y, A)$, making them $G$-spaces.
Scheme

- $X = \text{configuration space} = \text{space of all candidates}$

- $(Y, A) = \text{test space with subspace}$

- The symmetry of the problem induces an action of some group $G$ both on $X$ and $(Y, A)$, making them $G$-spaces.

- Test map $f : X \rightarrow (Y, A)$ which is a $G$-map.
Scheme

- $X =$ configuration space $=$ space of all candidates
- $(Y, A) =$ test space with subspace
- The symmetry of the problem induces an action of some group $G$ both on $X$ and $(Y, A)$, making them $G$-spaces.
- Test map $f : X \to (Y, A)$ which is a $G$-map.
- $x \in X$ is a solution to the problem iff $f(x) \in A$. 
Scheme, continued

- If we show that for every $G$-map $f$, $\text{im } f \cap A \neq \emptyset$, the problem has a solution.
If we show that for every $G$-map $f$, $\text{im } f \cap A \neq \emptyset$, the problem has a solution.

Most often $Y = \mathbb{R}^n$ and $A = \{0\}$ (a subspace or the subspace arrangement). In the former case the problem reduces to the question whether for a $G$-map $f : X \to \mathbb{R}^n$, the requirement $0 \in \text{im } f$ has to be satisfied.
If we show that for every $G$-map $f$, $\text{im } f \cap A \neq \emptyset$, the problem has a solution.

Most often $Y = \mathbb{R}^n$ and $A = \{0\}$ (a subspace or the subspace arrangement). In the former case the problem reduces to the question whether for a $G$-map $f : X \rightarrow \mathbb{R}^n$, the requirement $0 \in \text{im } f$ has to be satisfied.

Since $\mathbb{R}^n \setminus \{0\} \simeq S^{n-1}$, this is equivalent with the claim that there is no $G$-map $f : X \rightarrow S^{n-1}$. 
DOLD’S THEOREM: If the space $X$ is $n$-connected, the space $Y$ is $n$-dimensional and the group $G$ acts freely on these spaces, then there is no $G$-map from $X$ to $Y$. 
Methods I - Dold’s theorem

**Dold’s Theorem:** If the space $X$ is $n$-connected, the space $Y$ is $n$-dimensional and the group $G$ acts freely on these spaces, then there is no $G$-map from $X$ to $Y$.

Here, the space $X$ is $n$-connected if its homology groups $\tilde{H}_0(X), H_1(X), \ldots, H_n(X)$ and its fundamental group $\pi_1(X)$ are trivial.
Methods I - Dold’s theorem

- **Dold’s Theorem**: If the space $X$ is $n$-connected, the space $Y$ is $n$-dimensional and the group $G$ acts freely on these spaces, then there is no $G$-map from $X$ to $Y$.

- Here, the space $X$ is $n$-connected if its homology groups $\tilde{H}_0(X), H_1(X), ..., H_n(X)$ and its fundamental group $\pi_1(X)$ are trivial.

- The action of the group $G$ on the space $X$ is free if for any $x \in X$ and any $g \in G$ the equality $g \cdot x = x$ implies $g = e$, the neutral element of $G$. 

Equivariant methods – p. 10/4
Dold’s theorem, continued

The Dold’s theorem reduces, in many cases, the considered problem to the determination of connectivity of the configuration space.
Dold’s theorem, continued

- The Dold’s theorem reduces, in many cases, the considered problem to the determination of connectivity of the configuration space.

- For example, for the continuous Tverberg theorem, the configuration space was a cell complex whose cells were products of $p$ disjoint simplices of $\Delta^N$ (deleted product), and the key technical step in the proof was the establishing of high connectivity of this complex.
Dold’s theorem, continued

- Using deleted joins instead of deleted products, K. Sarkaria showed that the proof could be substantially simplified, but this could not be done in some other examples.
Dold’s theorem, continued

- Using deleted joins instead of deleted products, K. Sarkaria showed that the proof could be substantially simplified, but this could not be done in some other examples.

- In more complicated cases high connectivity of the configuration space could be established using the nerve lemma or some spectral sequence.
Using deleted joins instead of deleted products, K. Sarkaria showed that the proof could be substantially simplified, but this could not be done in some other examples.

In more complicated cases high connectivity of the configuration space could be established using the nerve lemma or some spectral sequence.

For the colored Tverberg theorem, the configuration space was the join of chessboard complexes and it was important to determine their connectivity.
Dold’s theorem, continued

- **Colored Tverberg theorem**: Given \( n + 1 \) collections of \( 2p - 1 \) points \( C_0, C_1, ..., C_n \) in \( \mathbb{R}^n \), there are \( p \) simplices with vertices of different colors, which intersect.
Dold’s theorem, continued

- **Colored Tverberg Theorem**: Given $n + 1$ collections of $2p - 1$ points $C_0, C_1, \ldots, C_n$ in $\mathbb{R}^n$, there are $p$ simplices with vertices of different colors, which intersect.

- Configuration space is
  \[
  ([2p - 1]^{*(d+1)})^p = ([2p - 1]^p)^{(d+1)}.
  \]
Dold’s theorem, continued

- **Colored Tverberg theorem:** Given \( n + 1 \) collections of \( 2p - 1 \) points \( C_0, C_1, \ldots, C_n \) in \( \mathbb{R}^n \), there are \( p \) simplices with vertices of different colors, which intersect.

- Configuration space is
  \[
  ([2p - 1]^{\delta(d+1)})^p = ([2p - 1]_{\delta}^p)^{(d+1)}.
  \]

- \([2p - 1]_{\delta}^p\) is an \( (p - 2)\)-connected chessboard complex. This object is important in Discrete and Computational Geometry, Representation Theory etc.
Dold’s theorem, continued

- Actually, we obtained more, the continuous version of the theorem.
Dold’s theorem, continued

- Actually, we obtained more, the continuous version of the theorem.

- Using the above, it is shown that the theorem is true for non-prime $p$ with $4p - 3$ points of each color.
Sometimes a vector space $Y$ is naturally assigned to every $x \in X$ in such way that we obtain the vector bundle over $X$. It is appropriate in such cases (instead of a $G$-map missing the origin) to consider nowhere-zero section of that bundle.
Methods II - characteristic classes

- Sometimes a vector space $Y$ is naturally assigned to every $x \in X$ in such way that we obtain the vector bundle over $X$. It is appropriate in such cases (instead of a $G$-map missing the origin) to consider nowhere-zero section of that bundle.

- It is well known that such section does not exist if certain characteristic class (Euler, Stiefel-Whitney) of that bundle does not vanish.
Characteristic classes, continued

The non-vanishing of some characteristic classes of the bundles over Grassmann manifold, Stiefel manifold,... could be established.
The non-vanishing of some characteristic classes of the bundles over Grassmann manifold, Stiefel manifold,... could be established.

A link between equivariant maps and sections of vector bundles is provided by:

**Theorem:** Let $X$ be a principal $G$-bundle over $B$ and $Y$ a $G$-vector space. Then the sections of the induced vector bundle $X \times_G Y$ are in 1-1 correspondence with the $G$-maps $f : X \rightarrow Y$. 

EQUIVARIANT METHODS – p. 16/4
Characteristic classes, continued

**Theorem:** Let $\mu_0, \mu_1, \ldots, \mu_k$ be a collection of $\sigma$-additive probability measures in $\mathbb{R}^n$. Then there exist a $k$-dimensional plane $F$ such that every closed halfspace $H$ containing $F$ satisfies $\mu_i(H) \geq \frac{1}{n-k+1}$ for every $i = 0, 1, \ldots, k$. 


Characteristic classes, continued

\begin{itemize}
\item \textbf{Theorem:} Let $\mu_0, \mu_1, \ldots, \mu_k$ be a collection of $\sigma$-additive probability measures in $\mathbb{R}^n$. Then there exist a $k$-dimensional plane $F$ such that every closed halfspace $H$ containing $F$ satisfies $\mu_i(H) \geq \frac{1}{n-k+1}$ for every $i = 0, 1, \ldots, k$.

\item $k = n - 1$ is the ham-sandwich theorem, and $k = 0$ is the Rado’s center-point theorem.
\end{itemize}
Theorem: Let \( \mu_0, \mu_1, \ldots, \mu_k \) be a collection of \( \sigma \)-additive probability measures in \( \mathbb{R}^n \). Then there exist a \( k \)-dimensional plane \( F \) such that every closed halfspace \( H \) containing \( F \) satisfies \( \mu_i(H) \geq \frac{1}{n-k+1} \) for every \( i = 0, 1, \ldots, k \).

\( k = n - 1 \) is the ham-sandwich theorem, and \( k = 0 \) is the Rado’s center-point theorem.

The configuration space is modelled by the intersection points \( x \) of \( k \)-planes \( F \) with their orthogonal complements \( L \in \text{Gr}_{n-k}(\mathbb{R}^n) \).
Characteristic classes, continued
Characteristic classes, continued

- So, the configuration space is the total space of the canonical bundle over the Grassmann manifold.
Characteristics classes, continued

- So, the configuration space is the total space of the canonical bundle over the Grassmann manifold.

- By projecting the measures to $(n - k)$-subspace $L$ and using the Rado’s theorem, we see that for every $L$ and every measure $\mu_i$ there is a point $x^L_i$ such that the plane $F_i$ containing $x^L_i$ orthogonal to $L$ satisfies the claim of the theorem for $\mu_i$. 
So, the configuration space is the total space of the canonical bundle over the Grassmann manifold.

By projecting the measures to \((n - k)\)-subspace \(L\) and using the Rado’s theorem, we see that for every \(L\) and every measure \(\mu_i\) there is a point \(x_i^L\) such that the plane \(F_i\) containing \(x_i^L\) orthogonal to \(L\) satisfies the claim of the theorem for \(\mu_i\).

The question is, whether for some \(L\) all these points coincide.
This is equivalent with the question whether for some $L$ the differences $x_1^L - x_0^L$, ..., $x_k^L - x_0^L$ all vanish.
This is equivalent with the question whether for some $L$ the differences $x^L_1 - x^L_0$, ..., $x^L_k - x^L_0$ all vanish.

For this, it is enough to prove that the Whitney sum $\gamma \oplus \cdots \oplus \gamma$ of $k$ copies of the canonical bundle over the Grassmann manifold $\text{Gr}_{n-k}(\mathbb{R}^n)$ does not admit a nowhere zero section.
This is equivalent with the question whether for some $L$ the differences $x^L_1 - x^L_0$, ..., $x^L_k - x^L_0$ all vanish.

For this, it is enough to prove that the Whitney sum $\gamma \oplus \cdots \oplus \gamma$ of $k$ copies of the canonical bundle over the Grassmann manifold $\text{Gr}_{n-k}(\mathbb{R}^n)$ does not admit a nowhere zero section.

Now, it suffices to prove that $w_n(\gamma)^k$ is a non-zero element in $H^*(\text{Gr}_{n-k}(\mathbb{R}^n); \mathbb{Z}/2)$. 

EQUIVARIANT METHODS – p. 20/4
This turns out to be equivalent with the following algebraic claim:

**Claim:** The monomial \((t_1 t_2 \cdots t_{n-k})^k\) does not belong to the ideal generated by the symmetric monomials in the variables \(s_1, \ldots, s_k, t_1, \ldots, t_{n-k}\).
This turns out to be equivalent with the following algebraic claim:

**Claim:** The monomial \((t_1 t_2 \cdots t_{n-k})^k\) does not belong to the ideal generated by the symmetric monomials in the variables \(s_1, \ldots, s_k, t_1, \ldots, t_{n-k}\).

There is a number of ways to prove this claim. We provided a topological proof using the ideal-valued cohomological index theory of Fadell and Husseini.
The universal $G$-bundle $EG \to BG$ consists of a contractible $G$-space $EG$ on which the group $G$ acts freely and the orbit space $BG = EG/G$. 
Methods III - index theory

- The universal $G$-bundle $EG \to BG$ consists of a contractible $G$-space $EG$ on which the group $G$ acts freely and the orbit space $BG = EG/G$.

- The equivariant map $f : X \to Y$ induces (after multiplying by $EG$ and taking the orbits of $G$ action) the commutative diagram:

\[
\begin{array}{ccc}
X \times_G EG & \xrightarrow{\tilde{f}} & Y \times_G EG \\
\downarrow p_1 & & \downarrow p_2 \\
BG & & \\
\end{array}
\]
Index theory, continued

In cohomology we have:

\[ H^*_G(X) \xrightarrow{f^*} H^*_G(Y) \]

\[ H^*(BG) \]

\[ H^*(BG) \]

\[ H^*_G(X) \]

\[ H^*_G(Y) \]

\[ H^*(BG) \]

\[ H^*(BG) \]

\[ H^*_G(X) \xleftarrow{p^*_1} \]

\[ H^*_G(Y) \xleftarrow{p^*_2} \]

Figure 4:
The kernels of $p_1^*$ and $p_2^*$ are the ideals in the cohomology ring of the classifying space $BG$ of the group $G$ and are called indices and denoted by $\text{Ind}_G(X)$ and $\text{Ind}_G(Y)$. 
Index theory, continued

- The kernels of $p_1^*$ and $p_2^*$ are the ideals in the cohomology ring of the classifying space $BG$ of the group $G$ and are called indices and denoted by $\text{Ind}_G(X)$ and $\text{Ind}_G(Y)$.

- From the above commutative diagram it follows $\text{Ind}_G(Y) \subseteq \text{Ind}_G(X)$. 
Index theory, continued

- The kernels of $p_1^*$ and $p_2^*$ are the ideals in the cohomology ring of the classifying space $BG$ of the group $G$ and are called indices and denoted by $\text{Ind}_G(X)$ and $\text{Ind}_G(Y)$.

- From the above commutative diagram it follows $\text{Ind}_G(Y) \subseteq \text{Ind}_G(X)$.

- If we could determine these ideals and show that this relation does not hold, the obtained contradiction would show that there is no equivariant map from $X$ to $Y$. 

EQUIVARIANT METHODS – p. 24/4
If $X$ is a CW-complex and the space $Y \setminus A$ is $(n - 1)$-connected, there is a $G$-map $f$ from $n$-skeleton of $X$ to $Y \setminus A$. 
If $X$ is a CW-complex and the space $Y \setminus A$ is $(n - 1)$-connected, there is a $G$-map $f$ from $n$-skeleton of $X$ to $Y \setminus A$.

To some $(n + 1)$-cell $e^{n+1}$ from every orbit, we assign the homotopy class of the composition of the characteristic map $\varphi e^{n+1} : S^n \to X_n$ of this cell with $f$ and extend this equivariantly.
If $X$ is a CW-complex and the space $Y \setminus A$ is $(n - 1)$-connected, there is a $G$-map $f$ from $n$-skeleton of $X$ to $Y \setminus A$.

To some $(n + 1)$-cell $e^{n+1}$ from every orbit, we assign the homotopy class of the composition of the characteristic map $\varphi_{e^{n+1}} : S^n \to X_n$ of this cell with $f$ and extend this equivariantly.

In such way we obtain cochain $o(f) \in C^{n+1}_G(X; \pi_n(Y))$. 
Obstruction theory, continued

This cochain is a cocycle and the map $f$ can be extended over the $(n + 1)$-skeleton of $X$ (after modifying on $n$-cells, but not on $(n - 1)$-skeleton of $X$) iff its cohomology class (called the obstruction class) equals zero.
Obstruction theory, continued

- This cochain is a cocycle and the map $f$ can be extended over the $(n + 1)$-skeleton of $X$ (after modifying on $n$-cells, but not on $(n - 1)$-skeleton of $X$) iff its cohomology class (called the obstruction class) equals zero.

- If $X$ and $Y$ are manifolds and we have a "generic" map $g : X \to Y$, the Poincaré dual to this class is represented by the fundamental class of the submanifold $g^{-1}(A)$ of $X$. So, it suffices to prove that this class does not vanish.
If the codimension of the $G$-invariant submanifold $A$ in the manifold $Y$ equals the dimension of the manifold $X$, the submanifold $g^{-1}(A)$ is a finite set, and it is enough to check that it contains an odd number of points.
Obstruction theory, continued

- If the codimension of the $G$-invariant submanifold $A$ in the manifold $Y$ equals the dimension of the manifold $X$, the submanifold $g^{-1}(A)$ is a finite set, and it is enough to check that it contains an odd number of points.

- The cases when $g^{-1}(A)$ is at least 1-dimensional are much more complicated.
Equipartition problem

In 1960, B. Grünbaum posed the following **QUESTION**: When do any $j$ measurable sets in $\mathbb{R}^d$ admit an equipartition by $k$ hyperplanes?
In 1960, B. Grünbaum posed the following question:

**Question:** When do any \( j \) measurable sets in \( \mathbb{R}^d \) admit an equipartition by \( k \) hyperplanes?

If this is the case we say that the triple \((d, j, k)\) is admissible. This question could be reformulated in the following way.
In 1960, B. Grünbaum posed the following **QUESTION**: When do any \( j \) measurable sets in \( \mathbb{R}^d \) admit an equipartition by \( k \) hyperplanes?

If this is the case we say that the triple \( (d, j, k) \) is admissible. This question could be reformulated in the following way.

**PROBLEM**: Determine \( \Delta(j, k) \), the smallest dimension \( d \) such that the triple \( (d, j, k) \) is admissible.
$k = 1$ is the ham-sandwich theorem. It says that $(d, d, 1)$ is admissible or $\Delta(d, 1) = d$. 
Equi-partition problem, continued

- $k = 1$ is the ham-sandwich theorem. It says that $(d, d, 1)$ is admissible or $\Delta(d, 1) = d$.

- What about the case $j = 1$ (one measurable set)?
$k = 1$ is the ham-sandwich theorem. It says that $(d, d, 1)$ is admissible or $\Delta(d, 1) = d$.

What about the case $j = 1$ (one measurable set)?

$\Delta(1, 2) = 2$ is simple and $\Delta(1, 3) = 3$ is obtained by H. Hadwiger in 1966. He also showed $\Delta(2, 2) = 3$ and $\Delta(2, 3) = 5$. 
E. Ramos, building on some previous results, introduced new ideas (a generalization of the Borsuk-Ulam theorem to the product of spheres with a combinatorial proof) and obtained some new and general results about the admissible triples or the function $d = \Delta(j, k)$. 
E. Ramos, building on some previous results, introduced new ideas (a generalization of the Borsuk-Ulam theorem to the product of spheres with a combinatorial proof) and obtained some new and general results about the admissible triples or the function \( d = \Delta(j, k) \).

By considering the system of \( j \) interval measures concentrated on the moment curve, he showed that for admissible triples the inequality \( dk \geq j(2^k - 1) \) has to be satisfied, or in other words \( \Delta(j, k) \geq \frac{j(2^k - 1)}{k} \).
Besides that, he also improved the upper bounds for the function $d = \Delta(j, k)$ when $j$ is a power of 2. He also proved e.g. that $\Delta(1, 4) \leq 5$, $\Delta(5, 2) \leq 9$ and $\Delta(3, 3) \leq 9$. 
Besides that, he also improved the upper bounds for the function $d = \Delta(j, k)$ when $j$ is a power of 2. He also proved e.g. that $\Delta(1, 4) \leq 5$, $\Delta(5, 2) \leq 9$ and $\Delta(3, 3) \leq 9$.

The only obtained general upper bound $\Delta(j, k) \leq j2^{k-1}$ is obtained by the successive application of the ham-sandwich theorem.
Besides that, he also improved the upper bounds for the function \( d = \Delta(j, k) \) when \( j \) is a power of 2. He also proved e.g. that \( \Delta(1, 4) \leq 5 \), \( \Delta(5, 2) \leq 9 \) and \( \Delta(3, 3) \leq 9 \).

The only obtained general upper bound \( \Delta(j, k) \leq j2^{k-1} \) is obtained by the successive application of the ham-sandwich theorem.

We have \( \frac{j(2^k-1)}{k} \leq \Delta(j, k) \leq j2^{k-1} \), and one is inclined to believe that the lower bound is tight.
The most interesting cases left open are to decide whether the triples \((4, 1, 4)\), \((8, 5, 2)\) and \((7, 3, 3)\) are admissible.
The most interesting cases left open are to decide whether the triples $(4, 1, 4)$, $(8, 5, 2)$ and $(7, 3, 3)$ are admissible.

Let me present, as the illustration, some results from

P. Mani, S. V. and R. Živaljević, Topology and combinatorics of partition of masses by hyperplanes, Adv. in Math., to appear
The most interesting cases left open are to decide whether the triples \((4, 1, 4)\), \((8, 5, 2)\) and \((7, 3, 3)\) are admissible.

Let me present, as the illustration, some results from

P. Mani, S. V. and R. Živaljević, Topology and combinatorics of partition of masses by hyperplanes, Adv. in Math., to appear

The geometrical question will be reduced to the topological one and the latter will be resolved using the combinatorial argument.
Configuration space

We embed $\mathbb{R}^d$ in $\mathbb{R}^{d+1}$ at the level 1, i.e.
$\mathbb{R}^d \approx \mathbb{R}^d \times \{1\} \hookrightarrow \mathbb{R}^{d+1}$. 
Configuration space

- We embed $\mathbb{R}^d$ in $\mathbb{R}^{d+1}$ at the level 1, i.e.
  $\mathbb{R}^d \approx \mathbb{R}^d \times \{1\} \hookrightarrow \mathbb{R}^{d+1}$.

- an oriented hyperplane $H$ in $\mathbb{R}^d \times \{1\}$ ↔ an oriented $d$-subspace $L_H$ in $\mathbb{R}^{d+1}$ ↔ a vector $x_H$ in $S^d$
Configuration space

- We embed $\mathbb{R}^d$ in $\mathbb{R}^{d+1}$ at the level 1, i.e.
  $\mathbb{R}^d \approx \mathbb{R}^d \times \{1\} \rightarrow \mathbb{R}^{d+1}$.

- an oriented hyperplane $H$ in $\mathbb{R}^d \times \{1\}$ ↔
an oriented $d$-subspace $L_H$ in $\mathbb{R}^{d+1}$ ↔
a vector $x_H$ in $S^d$

- So, the configuration space is
  $S^d \times \cdots \times S^d = (S^d)^k$. 
Configuration space

$R^d \times \{1\}$

Figure 5:
Test space and test map

- $k$ oriented $d$-subspaces in $\mathbb{R}^{d+1}$ determine $2^k$ hyperorthants.
Test space and test map

- $k$ oriented $d$-subspaces in $\mathbb{R}^{d+1}$ determine $2^k$ hyperorthants.

- For every measurable set, the measures of its intersections with these hyperorthants sum up to 1.
Test space and test map

- $k$ oriented $d$-subspaces in $\mathbb{R}^{d+1}$ determine $2^k$ hyperorthants.

- For every measurable set, the measures of its intersections with these hyperorthants sum up to 1.

- If we denote $V = \{ v \in \mathbb{R}^{2^k} \mid \sum_i v_i = 1 \}$, the test space is $Y = V^j$, and the invariant subspace is a one-point set $A = \{ (\frac{1}{2^k}, \ldots, \frac{1}{2^k}) \}$
Test space and test map

- $k$ oriented $d$-subspaces in $\mathbb{R}^{d+1}$ determine $2^k$ hyperorthants.

- For every measurable set, the measures of its intersections with these hyperorthants sum up to 1.

- If we denote $V = \{ v \in \mathbb{R}^{2^k} \mid \sum_i v_i = 1 \}$, the test space is $Y = V^j$, and the invariant subspace is a one-point set $A = \{ (\frac{1}{2^k}, \ldots, \frac{1}{2^k}) \}$

- Trivially $Y \setminus A \simeq S^j(2^k-1)-1$. 

EquiVARIANT METHODS – p. 35/4
The group of symmetries is the Weyl group $W_k = (\mathbb{Z}/2)^k \rtimes S_k$, the symmetry group of a $k$-dimensional cube.
Group of symmetries

- The group of symmetries is the Weyl group $W_k = (\mathbb{Z}/2)^\oplus k \rtimes S_k$, the symmetry group of a $k$-dimensional cube.

- The involutions from $\mathbb{Z}/2$ send the corresponding coordinates to their antipodal points in the configuration space, and the elements of the symmetric group permute the coordinates. All elements permute the above hyperorthants and so act on the test space by the permutations.
Group of symmetries

The group of symmetries is the Weyl group $W_k = (\mathbb{Z}/2)^{\oplus k} \rtimes S_k$, the symmetry group of a $k$-dimensional cube.

The involutions from $\mathbb{Z}/2$ send the corresponding coordinates to their antipodal points in the configuration space, and the elements of the symmetric group permute the coordinates. All elements permute the above hyperorthants and so act on the test space by the permutations.

In the case of interest in this lecture $k = 2$, we get a dihedral group $D_8$. 

EQUIVARIANT METHODS – p. 36/4
The case $k = 2$

- We obtained the explicit combinatorial algorithm which in terms of some combinatorial functions decides if a $\mathbb{D}_8$-map exists, when treating the triples of the form $(6m - 1, 4m - 1, 2)$ and $(6m + 2, 4m + 1, 2)$.
The case $k = 2$

- We obtained the explicit combinatorial algorithm which in terms of some combinatorial functions decides if a $\mathbb{D}_8$-map exists, when treating the triples of the form $(6m - 1, 4m - 1, 2)$ and $(6m + 2, 4m + 1, 2)$.

- We restrict our attention here to the case of the triple $(8, 5, 2)$.
The case $k = 2$

- We obtained the explicit combinatorial algorithm which in terms of some combinatorial functions decides if a $\mathbb{D}_8$-map exists, when treating the triples of the form $(6m - 1, 4m - 1, 2)$ and $(6m + 2, 4m + 1, 2)$.

- We restrict our attention here to the case of the triple $(8, 5, 2)$.

- We use the obstruction theory approach first, and for a generic map $g$ take the test map induced by the 5 intervals placed along the moment curve in $\mathbb{R}^8$. 
$(8, 5, 2)$ is admissible

It turns out that the Poincaré dual of the obstruction class is the fundamental class of the union of three circles in $H_1^{D8}(S^8 \times S^8; \mathbb{Z})$. 
(8, 5, 2) \textit{is admissible}

- It turns out that the Poincaré dual of the obstruction class is the fundamental class of the union of three circles in $H^1_{D^8}(S^8 \times S^8; \mathbb{Z})$.

- Since $S^8 \times S^8$ is 7-connected, $H^1_{\mathbb{D}^8}(S^8 \times S^8; \mathbb{Z}) \cong H_1(\mathbb{D}^8; \mathbb{Z})$. 
(8, 5, 2) is admissible

- It turns out that the Poincaré dual of the obstruction class is the fundamental class of the union of three circles in $H_1^{\mathbb{D}_8}(S^8 \times S^8; \mathbb{Z})$.

- Since $S^8 \times S^8$ is 7-connected, $H_1^{\mathbb{D}_8}(S^8 \times S^8; \mathbb{Z}) \cong H_1(\mathbb{D}_8; \mathbb{Z})$.

- By representing the obtained three circles by the sequences of the signed words in two letters, and by identifying the latter as the elements of the homology of the dihedral group, we get that the obstruction class is non-trivial.
Identification of the singular set

Figure 6: Metamorphoses of curves
(8, 5, 2) is admissible

So, there is no equivariant map, and the triple (8, 5, 2) is admissible, or $\Delta(5, 2) = 8$. 
(8, 5, 2) is admissible

So, there is no equivariant map, and the triple (8, 5, 2) is admissible, or \( \Delta(5, 2) = 8 \).

For the triple (4, 1, 4), R. Živaljević (TAMS, to appear) showed that an equivariant map does exist, and so this approach does not provide an answer to the original question.
(8, 5, 2) is admissible

- So, there is no equivariant map, and the triple (8, 5, 2) is admissible, or $\Delta(5, 2) = 8$.

- For the triple (4, 1, 4), R. Živaljević (TAMS, to appear) showed that an equivariant map does exist, and so this approach does not provide an answer to the original question.

- The same is very likely to be true for the triple (7, 3, 3).
Index theory approach

In this approach we work with the subgroup $H = (\mathbb{Z}/2)^k$ of the symmetry group $W_k$. 
Index theory approach

- In this approach we work with the subgroup \( H = (\mathbb{Z}/2)^\oplus k \) of the symmetry group \( W_k \).

- We know \( H^*(BH; \mathbb{F}_2) = \mathbb{F}_2[x_1, \ldots, x_k] \) and the indices are \( \text{Ind}_H((S^d)^k) = (x_1^{d+1}, \ldots, x_k^{d+1}) \) and \( \text{Ind}_H(S(Y)) = ((\mathbb{P}_k(x_1, \ldots, x_k))^j) \), where \( \mathbb{P}_k(x_1, \ldots, x_k) = x_1 \cdots x_k (x_1 + x_2) \cdots (x_{k-1} + x_k) \cdots (x_1 + \cdots + x_k) \) is a Dickson polynomial.
Index theory approach

- In this approach we work with the subgroup $H = (\mathbb{Z}/2)^{\oplus k}$ of the symmetry group $W_k$.

- We know $H^*(BH; \mathbb{F}_2) = \mathbb{F}_2[x_1, \ldots, x_k]$ and the indices are $\text{Ind}_H((S^d)^k) = (x_1^{d+1}, \ldots, x_k^{d+1})$ and $\text{Ind}_H(S(Y)) = ((\mathbb{P}_k(x_1, \ldots, x_k))^j)$, where

\[ \mathbb{P}_k(x_1, \ldots, x_k) = x_1 \cdots x_k(x_1 + x_2) \cdots (x_{k-1} + x_k) \cdots (x_1 + \cdots + x_k) \]

is a Dickson polynomial.

- Over $\mathbb{F}_2$, a Dickson polynomial has also another description and as a consequence we get:
**THEOREM:** Let

\[ P_k = \text{Det} \begin{bmatrix}
    x_1 & x_1^2 & x_1^4 & \ldots & x_1^{2^{k-1}} \\
    x_2 & x_2^2 & x_2^4 & \ldots & x_2^{2^{k-1}} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    x_k & x_k^2 & x_k^4 & \ldots & x_k^{2^{k-1}}
\end{bmatrix} \in \mathbb{F}_2[x_1, \ldots, x_k]

be a Dickson polynomial. Then \( j \) measures in \( R^d \) admit an equipartition by \( k \) hyperplanes if

\[(P_k)^j \notin \text{Ideal}\{x_1^{d+1}, \ldots, x_k^{d+1}\}\]
Let \( j = 2^q + r, \ 0 \leq r \leq 2^q - 1 \). Then the above considerations give \( \Delta(2^q + r, k) \leq 2^{k+q-1} + r \).
Index theory approach

- Let $j = 2^q + r$, $0 \leq r \leq 2^q - 1$. Then the above considerations give $\Delta(2^q + r, k) \leq 2^{k+q-1} + r$.

- This upper estimate is strictly better than the previous one if $r \neq 0$ and equals the previous one if $r = 0$. 
Let \( j = 2^q + r, \) \( 0 \leq r \leq 2^q - 1. \) Then the above considerations give \( \Delta(2^q + r, k) \leq 2^{k+q-1} + r. \)

This upper estimate is strictly better than the previous one if \( r \neq 0 \) and equals the previous one if \( r = 0. \)

As a consequence we get
\[
\Delta(2^{q+1} - 1, 2) = 3 \cdot 2^q - 1,
\]
\[
\Delta(2^{q+1} - 2, 2) = 3 \cdot 2^q - 3 + \varepsilon \text{ and}
\]
\[
\Delta(2^{q+1} - 3, 2) = 3 \cdot 2^q - 4 + \eta, \text{ where}
\]
\( \varepsilon, \eta \in \{0, 1\}. \)
Index theory approach

Here are some particular examples which illustrate the power of the previous theorem. The best previously known upper bounds are given in parentheses.

\[ 17 \leq \Delta(7, 3) \leq 19 \quad (28), \]
\[ 14 \leq \Delta(6, 3) \leq 18 \quad (24), \]
\[ 35 \leq \Delta(15, 3) \leq 39 \quad (60), \]
\[ 33 \leq \Delta(14, 3) \leq 38 \quad (56), \]
\[ 27 \leq \Delta(7, 4) \leq 35 \quad (56), \]
\[ 23 \leq \Delta(6, 4) \leq 34 \quad (48), \]
\[ 57 \leq \Delta(15, 4) \leq 71 \quad (120), \]
\[ 53 \leq \Delta(14, 4) \leq 70 \quad (112), \]
THANK YOU FOR YOUR ATTENTION!