Nastava matematike - od Velike škole do danas

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„Međutim potvrđuje se sve više da su njegovi rezultati iz teorije polinoma, diferencijalnih jednačina polazna tačka za razne generalizacije. Ostaje da se ispitaju dubina i uticaj Petrovićevih rezultata u svetskoj matematici."

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- i1) if $f'(a) > 0$, then f is increasing;
- i2) if $f'(a) < 0$, then f is decreasing.

Mika uses the formula $f'(a) = \dfrac{f(a+h) - f(a)}{h}$. This is true only for functions of the form $f(x) = f(a) + f'(a)(x - a)$.

Further, if $f'(a) > 0$ by definition of the limit

$$\frac{f(a+h) - f(a)}{h} > 0$$

for h small enough (there is $h_0 > 0$, $|h| \leq h_0$) and therefore

$$f(a+h) - f(a) > 0,$$

for $0 < h < h_0$ and

$$f(a+h) - f(a) < 0,$$

for $-h_0 < h < 0$.

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To get a rigorous proof we can use that $f'(a)$ is limit of $\frac{f(a+h) - f(a)}{h}$ when h tends 0.

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If we divide an interval $[a, b]$ by n points $a < x_1 < x_2 < \ldots < x_n < b$ we have $n + 1$ interval $ax_1, x_1x_2, \ldots, x_nb$.

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In particular, we conclude

(A) if number of intervals is even, the number of points is odd.

(A1) Suppose that $f > 0$ on (a, b), $f(a) = f(b) = 0$ and that there is $a = x_0 < x_1 < \ldots < x_n < b$ such that f is increasing (decreasing) on $I_k = [x_k, x_{k+1}]$ for k even (odd).

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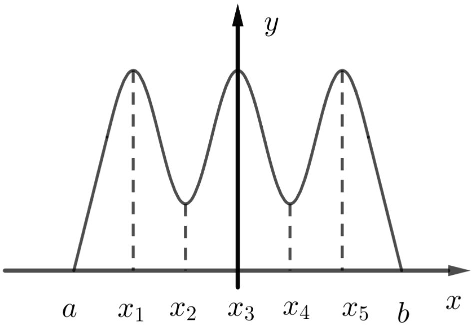
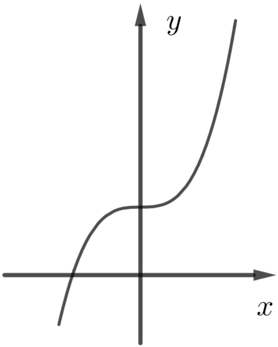
Interpret (A1) by sequence $XYXY \cdots XY$.

Under assumption (A1) between two consecutive (successive) local maximum there is a local minimum. Hence the total number of both is odd. The number of intervals is even and by (A) the number of points is odd.

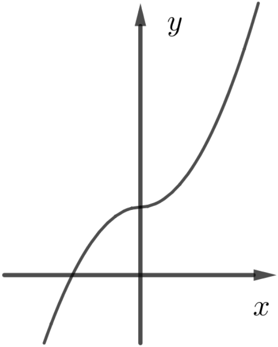


y
a
x_1
x_2
x_3
x_4
x_5
b
x

But this argument does not count the zero of f' if $f(x) = x^3 + 1$ or $f(x) = \operatorname{sgn} x \cdot x^2 + 1$ at 0.



$y = x^3 + 1$



$y = \operatorname{sgn} x \cdot x^2 + 1$

Statement 2

Suppose that f is continuous on $[a, b]$ and has derivative on (a, b). If $f > 0$ on (a, b) and f' has finite number of zeros (say $m-$zeros) on (a, b) and at every zeros of f', f has a local extreme, then m is odd. Roughly speaking, between two consecutive (successive) zeros of a function f there is odd number of zeros of function f'.

Mika gives application to geometric problems.

Example 1

In isosceles (in particular equilateral) triangle Δ describe rectangle R of maximal area.

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In isosceles (in particular equilateral) triangle Δ describe rectangle R of maximal area.

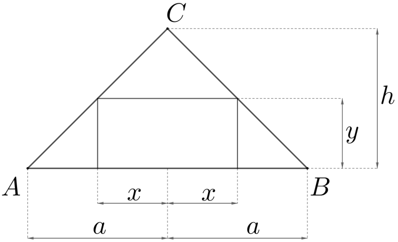
Let $AB = 2a$ be a base and h altitude of Δ and $2x$ (on AB) and y sides of R. Then

$$(a-x) : y = a : h$$

and therefore

$$P(x) = \frac{2h}{a} x(a-x).$$

Hence we find $x = a/2, y = h/2$.



Example 2

In a regular cone describe the cylinder of maximal volume.

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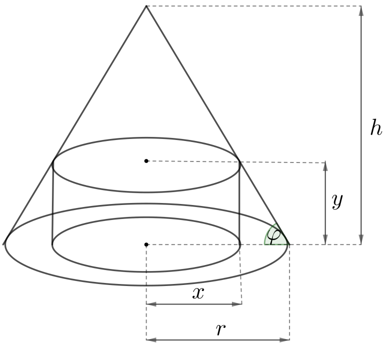
Let h be height of the cone and let r be radius of the cone's base. Denote by x and y radius and height of cylinder respectively. Then $\operatorname{tg}\varphi = \frac{h}{r}$ and hence

$$V(x) = \pi x^2 y = \pi \frac{h}{r} x^2 (r - x).$$

From $V'(x) = 0$, we find $x = \frac{2r}{3}$.



Испитивање тока функције помоћу извода

Узмимо функцију

$$y = F(x)$$

и пустимо да x расте почев од вредности a, онда се може десити да и сама функција почне да расте или опада.

Ако се образује извод $F'(x)$ па се у њему смени x са a, онда ако је добивени резултат позитиван, функција сигурно расте, кад само x почне расти од a па навише. Ако је резултат негативан, функција насигурно опада. Да би то доказали, уочимо извод $F'(x)$ и сменимо у њему x са a, па добијамо

$$F'(a) = \frac{F(a+h) - F(a)}{h} \quad 1.)$$

Ако функција расте, очевидно је да је

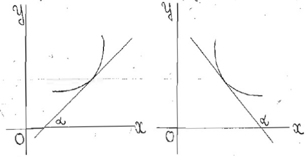
$$F(a+h) > F(a)$$

па су према томе и бројитељ и именитељ у једначини 1) позитивни и обратно ако је $F'(a)$ позитивно, мора и разлика у бројитељу бити позитивна. Ако је $F'(a)$ негативно, па је разлика такође негативна или

$$F(a+h) < F(a)$$

што значи да функција опада.

О томе се уверавамо и геометриски: Очевидно је из слике да



ако се крива пење, дирка гради оштар угао и према томе извод дате функције је позитиван. Ако крива линија силази, угао α је туп и према томе коефицијенат правца дирке т.ј. извод је негативан.

Rolle-ова теорема

У овој је теореми описана веза која постоји између вредности x-а, које поништавају једну дату функцију и вредности x-а које поништавају њен извод. Теорема се састоји у овоме:

Кад имамо једну функцију и ставимо да је она равна нули, добија се једначина

$$F(x)=0 \qquad 1.)$$

Под коренима ове једначине разумеју се оне вредности x-а које ту једначину задовољавају. Између два узастопна стварна корена једначине 1.) увек се налази непаран број корена изводне једначине

$$F'(x)=0 \qquad 2.)$$

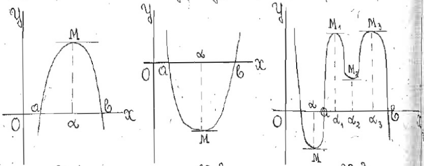
и обратно: између два узастопна стварна корена изводне једначине 2) може се налазити или само један или ни један корен једначине 1).

То је Rolle-ова теорема. Најлакше ћемо је доказати геометриски.

Ако конструишемо криву линију

$$y = F(x)$$

онда корени једначине 1) јесу апсцисе оних тачака у којима та крива сече

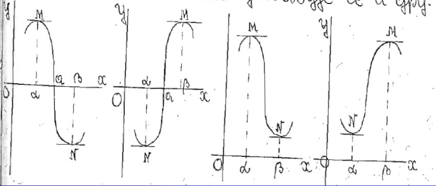


сл. 1. сл. 2. сл. 3.

апсцисну осовину. Означимо са a и b два узастопна корена једначине 1), онда ће у размаку од $x=a$ до $x=b$ крива линија имати један од облика сл. 1., 2. или 3. т.ј. крива може имати или само један

максимум и минимум као у сл. 1. и 2. или неколико као у сл. 3. Код сваког максимума или минимума дирка је паралелна апсцисној осовини и према томе њен коефицијенат правца $F'(x)$ је раван нули, што значи да између a и b мора постојати бар једна вредност x_α за коју ће бити $F'(x)=0$. У сл. 1. и 2. имамо једну такву вредност α у сл. 3. три. Број максимума и минимума увек је непаран број. Овим је доказан први део теореме т.ј. да се између два узастопна корена једначине 1) налази увек непаран број корена једначине 2).

Тако исто доказује се и дру-



III део теореме: Нека су α и β два узастопна стварна корена изводне једначине 2). Крива линија

$$y = F'(x)$$

имаће један од средњих облика.

тим функције

$$F(x) = x(32-x)$$

јер је то површина траженог правоугаоника. Први извод је

$$F(x) = -x + 32 - x$$

и он стављен раван нули даје

$$x = 16$$

Други извод је

$$F''(x) = -2$$

што значи да је та површина одиста највећа ако је једна страна правоугаониковa начињена од 16 палидрваца - т.ј. правоугаоник се своди на квадрат.

21. У равнокраки троугао $(2a, h)$ уписати један правоугаоник највеће површине.

Овди је

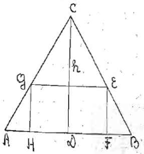
$$AB = 2a$$

$$CD = h$$

па ако означимо

$$DF = x$$

$$EF = y$$



онда из сличности троуглова BCD и BEF имамо

$$(a-x) : y = a : h$$

одакле

$$y = \frac{h(a-x)}{a}$$

па ће површина траженог правоугаоника бити, а то ће уједно бити и функција чији maximum тражимо,

$$F(x) = 2xy = \frac{2h}{a}x(a-x)$$

Њен први извод је

$$F'(x) = \frac{2h}{a}(a-x-x)$$

и он уједначен са нулом даје једначину

$$a - 2x = 0$$

одакле је

$$x = \frac{a}{2}$$

Отуда је

$$y = \frac{h(a-\frac{a}{2})}{a} = \frac{h}{2}$$

Како је

$$F''(x) = -2\frac{2h}{a}$$

то је дакле заиста површина уписаног правоугаоника највећа ако је
$x = \frac{a}{2}$ $y = \frac{h}{2}$.

$$\frac{h}{2}=\sqrt{a^2-x^2}$$

па је зато

$$V = x^2\pi\cdot h = 2x^2\pi\sqrt{a^2-x^2}$$

Одатле је

$$V' = 2\pi\frac{2a^2x-3x^3}{\sqrt{a^2-x^2}}$$

па

$$V'=0$$

даје једначину

$$2a^2x-3x^3=0$$

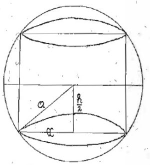
одакле је

$$x=\frac{a}{3}\sqrt{6}$$

Како је

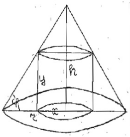
$$V''=2\pi\frac{2a^4-9a^2x^2+6x^4}{(a^2-x^2)^{3/2}}\Big|_{x=\frac{a}{3}\sqrt{6}} = -$$

т.ј. V је одиста maximum за $x=\frac{a}{3}\sqrt{6}$.



31. У праву кружну купу уписати прав цилиндар највеће запремине.

Ако је радиус основе купе r, њена висина h, радиус облице x а висина њена y, онда је, као што се види из слике



$$y = (r-x)\,\mathrm{tg}\,\varphi = \frac{h}{r}(r-x)$$

па је зато запремина облице

$$V = x^2\pi \cdot y = \pi \frac{h}{r} x^2 (r-x)$$

Одатле је

$$V' = \pi \frac{h}{r}(2rx - 3x^2)$$

па $V' = 0$ даје једначину

$$2rx - 3x^2 = 0$$

одакле је

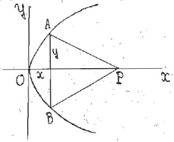
$$x = \frac{2}{3} r$$

Како је

$$V'' = \pi \frac{h}{r}(2r - 6x)_{x=\frac{2}{3}r} = -2\pi h$$

т.ј. запремина уписане облице одиста је maximum за $x = \frac{2}{3} r$.

32. У параболи $y^2 - 2px = 0$ уписати равнокраки троугао PAB (AB паралелно y-ској осовини) највеће површине (PO=a).



Површина уписаног троугла је

$$F = \frac{2y(a-x)}{2} = y(a-x)$$

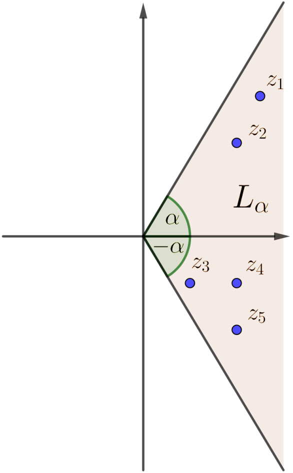
а како је из једначине параболе

$$y = \sqrt{2px}$$

Nejednakost Petrovića, Karamate, Smirnova i Kolmogorova

Neka je $L_\alpha = \{re^{it} : r > 0, |t| \le \alpha\}$ i $z_k = r_k e^{i\theta_k} \in L_\alpha$, $k = 1, ..., n$. U ovom paragrafu pretpostavljamo da je $\alpha \in (0, \pi/2)$.

Dakle, (A1): $z \in L_\alpha$
akko $\theta = \arg(z) \in [-\alpha, \alpha]$
akko $\cos(\theta) \ge \cos(\alpha)$
akko $\mathrm{Re} z \ge \cos(\alpha)|z|$.



Proposition 1 (Petrović)

Ako je $\alpha \in (0, \pi/2)$, *onda je*

$$\sum_{k=1}^{n} |z_k| \leq \frac{1}{\cos \alpha} \sum_{k=1}^{n} \mathrm{Re} z_k .$$

Proposition 2 (Karamata)

Ako $\alpha \in (0, \pi/2)$ *i ako je* $f : [a, b] \to L_\alpha$ *neprekidna funkcija* (*opštije integrabilna funkcija*), *onda je*

$$\int_a^b |f(t)|dt \leq \frac{1}{\cos\alpha} \mathrm{Re} \int_a^b f(t)dt.$$

Na osnovu (A1): $\mathrm{Re} f(t) \geq (\cos\alpha))|f(t)|$, $t \in [a, b]$ i otuda rezultat sledi.

Prema Mitrinoviću (D.S. Mitrinović, *O jednoj nejednakosti*, Mat. Biblioteka 38(1968), 93-96), propozicija 1 prvi put se pojavila u Petrovićevom članku u slučaju $\alpha = \pi/4$, 1917. godine. Petrović je dokazao opšti slučaj tek 1933. godine. Propozicija 2 je navedena u knjizi Jovana Karamate *Kompleksan broj sa primenom na elementarnu geometriju* iz 1950. godine na strani 157.

Jedna verzija Propozicije 2 koja se pripisuje Smirnovu i Kolmogorovu koristi se za dokaz sledećeg stava (videti I. I. Privalov, *Boundary Properties of Analytic Functions*,1950, str. 93):

Proposition 3 (Smirnov & Kolmogorov)

Ako je $f : \mathbb{B} \to L_\alpha$ *holomorfna, tada* $f \in H^p$, *gde je* $\alpha_0 = p\alpha < \pi/2$.

Skica dokaza. Neka je F grana funkcije f^p. Kako $F : \mathbb{B} \to L_{\theta_0}$ sledi

$$\int_0^{2\pi} |F(t)|dt \leq \frac{1}{\cos(\alpha_0)} \mathrm{Re} F(0).$$

Finsler type function F on $\mathbb{S}(a,b)$, $-\infty < a < b \leq \infty$, is defined for all tangent vectors $\mathbf{v} \in T_w$ by $F(\mathbf{v}) = F(w,\mathbf{v}) = \mathrm{Hyp}_{\mathbb{S}(a,b)}(w)|(v,e_1(w))|$. In particular,

$$F(v) = \frac{|(v,e_1(w))|}{\mathrm{Re}w - a} \quad \text{on} \quad \mathbb{S}(a,\infty).$$

Using Finsler type function F on $\mathbb{S}(a,b)$, (I1) can be stated as
(I2) $F(h^*) \leq |h|_{\mathrm{hyp}}$, where $h^* = df_z(h)$. If γ is a curve in $\mathbb{S}$ and suppose
(i): there is $\alpha \in [0,\pi/2)$ such that $\arg \gamma'(t) \leq \alpha$,
then $\cos\alpha\, |\gamma|_F \leq |\gamma|_{\mathrm{hyp}}$.

Proposition 4 (BG seminar 2017)

Suppose that D is a hyperbolic plane domain and $G = \mathbb{S}(a, b)$, $-\infty < a < b \leq \infty$, and $f : D \to G$ is a complex harmonic on the domain D. If $z_1, z_2 \in D$ and γ is geodesic arc join z_1 and z_2 and $\Gamma := f(\gamma)$ satisfies (i)*: $\Gamma' \in L_\alpha$, then*

$$\cos\alpha\, d_{\mathrm{hyp},G}(fz_1, fz_2) \leq d_{\mathrm{hyp},D}(z_1, z_2).$$

Pythagorean Theorem, Euclidean and non-Euclidean geometry and Time Dilation

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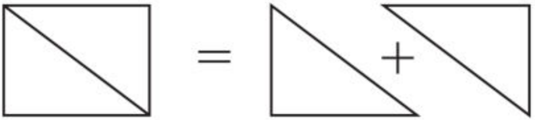
Srpska akademija nauka i umetnosti

Einstein's First Proof of Pythagorean Theorem

The Pythagorean theorem is true for rectangles of any proportion-skinny, blocky, or anything in between. The squares on the two sides always add up to the square on the diagonal. (More precisely, the areas of the squares, not the squares themselves, add up. But this simpler phrasing is less of a mouthful, so I'll continue to speak of squares adding up when I really mean their areas.) The same rule applies to right triangles, the shape you get when you slice a rectangle in half along its diagonal.



The rule now sounds more like the one you learned in school

$$a^2 + b^2 = c^2.$$

In pictorial terms, the squares on the sides of a right triangle add up to the square on its hypotenuse.

But why is the theorem true? What's the logic behind it? Actually, hundreds of proofs are known today. There's a marvelously simple one attributed to the Pythagoreans and, independently, to the ancient Chinese. There's an intricate one given in Euclid's Elements, which schoolchildren have struggled with for the past twenty-three hundred years, and which induced in the philosopher Arthur Schopenhauer "the same uncomfortable feeling that we experience after a juggling trick." There's even a proof by President James A. Garfield, which involves the cunning use of a trapezoid.

It helps to run through the proof quickly at first, to get a feel for its over-all structure.

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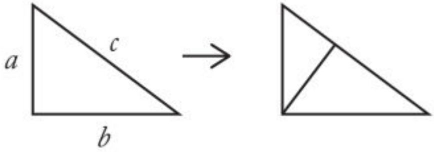
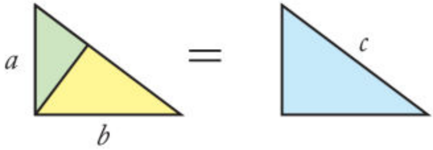
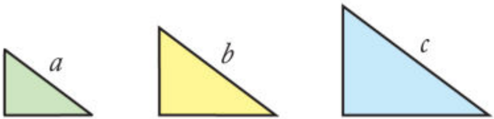
Step 1. Draw a perpendicular line from the hypotenuse to the right angle. This partitions the original right triangle into two smaller right triangles.



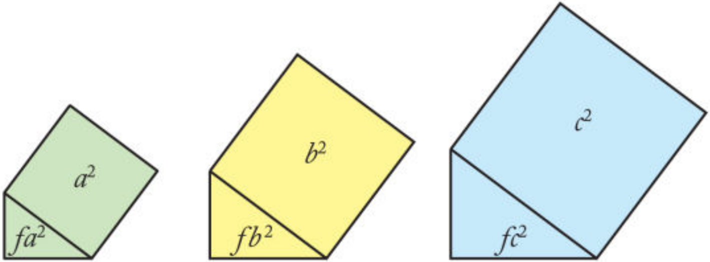
Step 2. Note that the area of the little triangle plus the area of the medium triangle equals the area of the big triangle.



Step 3. The big, medium, and little triangles are similar in the technical sense: their corresponding angles are equal and their corresponding sides are in proportion. Their similarity becomes clear if you imagine picking them up, rotating them, and arranging them like so, with their hypotenuses on the top and their right angles on the lower left:



Step 4. Because the triangles are similar, each occupies the same fraction f of the area of the square on its hypotenuse. Restated symbolically, this observation says that the triangles have areas fa^2, fb^2, and fc^2, as indicated in the diagram.



(Don't worry if this step provokes a bit of head-scratching. I'll have more to say about it below, after which I hope it'll seem obvious.)

Step 5. Remember, from Step 2, that the little and medium triangles add up to the original big one. Hence, from Step 4,

$$fa^2 + fb^2 = fc^2.$$

Step 6. Divide both sides of the equation above by f. You will obtain

$$a^2 + b^2 = c^2,$$

which says that the areas of the squares add up. That's the Pythagorean theorem.

The proof relies on two insights. The first is that a right triangle can be decomposed into two smaller copies of itself (Steps 1 and 3). That's a peculiarity of right triangles. If you try instead, for example, to decompose an equilateral triangle into two smaller equilateral triangles, you'll find that you can't. So Einstein's proof reveals why the Pythagorean theorem applies only to right triangles: they're the only kind made up of smaller copies of themselves. The second insight is about additivity. Why do the squares add up (Step 6)? It's because the triangles add up (Step 2), and the squares are proportional to the triangles (Step 4).

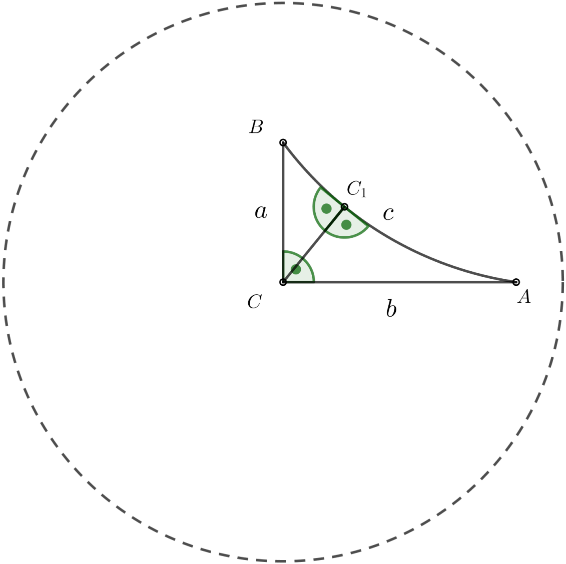
The logical link between the squares and triangles comes via the confusing Step 4. Here's a way to make peace with it. Try it out for the easiest kind of right triangle, an isosceles right triangle, also known as a 45-45-90 triangle, which is formed by cutting a square in half along its diagonal.

Many years after his Pythagorean proof, Einstein shared this lesson with another twelve-year-old who was wrestling with mathematics. On January 3, 1943, a junior-high-school student named Barbara Lee Wilson wrote to him for advice. “Most of the girls in my room have heroes which they write fan mail to,” she began. “You + my uncle who is in the Coast Guard are my heroes.” Wilson told Einstein that she was anxious about her performance in math class: “I have to work longer in it than most of my friends. I worry (perhaps too much).” Four days later, Einstein sent her a reply. “Until now I never dreamed to be something like a hero,” he wrote. “But since you have given me the nomination I feel that I am one.” As for Wilson’s academic concerns? “Do not worry about your difficulties in mathematics,” he told her. “I can assure you that mine are still greater.”

For a right triangle in hyperbolic geometry with sides a, b, c and with side c opposite a right angle, the relation between the sides takes the form:

$$\cosh c = \cosh a \ \cosh b$$

where cosh is the hyperbolic cosine.


B
C1
a
c
C
b
A

Last formula is a special form of the hyperbolic law of cosines that applies to all hyperbolic triangles:

$$\cosh c = \cosh a \, \cosh b - \sinh a \, \sinh b \, \cos \gamma$$

with γ the angle at the vertex opposite the side c.

By using the Maclaurin series for the hyperbolic cosine,

$$\cosh x = 1 + x^2/2 + o(x^2), \text{ when } x \to 0,$$

it can be shown that as a hyperbolic triangle becomes very small (that is, as a, b, and c all approach zero), the hyperbolic relation for a right triangle approaches the form of Pythagoras' theorem.

In hyperbolic geometry when the curvature is -1, the law of sines becomes:

$$\frac{\sin \alpha}{\sinh a} = \frac{\sin \beta}{\sinh b} = \frac{\sin \gamma}{\sinh c}.$$

In the special case when γ is a right angle, one gets

$$\sin \alpha = \frac{\sinh a}{\sinh c},$$

which is the analog of the formula in Euclidean geometry expressing the sine of an angle as the opposite side divided by the hypotenuse.

Professor R. Smullyan in his book 5000 B.C. and Other Philosophical Fantasies tells of an experiment he ran in one of his geometry classes. He drew a right triangle on the board with squares on the hypotenuse and legs and observed the fact the the square on the hypotenuse had a larger area than either of the other two squares. Then he asked, "Suppose these three squares were made of beaten gold, and you were offered either the one large square or the two small squares. Which would you choose?" Interestingly enough, about half the class opted for the one large square and half for the two small squares. Both groups were equally amazed when told that it would make no difference.

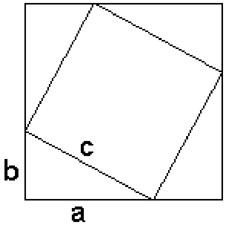
The Pythagorean (or Pythagoras') Theorem is the statement that the sum of (the areas of) the two small squares equals (the area of) the big one.

In algebraic terms, $a^2 + b^2 = c^2$ where c is the hypotenuse while a and b are the legs of the triangle.

The theorem is of fundamental importance in Euclidean Geometry where it serves as a basis for the definition of distance between two points. It's so basic and well known that, I believe, anyone who took geometry classes in high school couldn't fail to remember it long after other math notions got thoroughly forgotten.

The fourth approach starts with the same four triangles, except that, this time, they combine to form a square with the side $(a+b)$ and a hole with the side c. We can compute the area of the big square in two ways. Thus

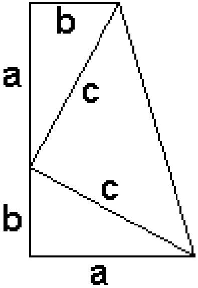
$$(a+b)^2 = 4ab/2 + c^2$$



simplifying which we get the needed identity.

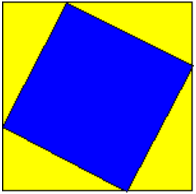
A proof which combines this with proof 3 is credited to the 12th century Hindu mathematician Bhaskara (Bhaskara II).

This proof, discovered by President J. A. Garfield in 1876 [Pappas], is a variation on the previous one. But this time we draw no squares at all. The key now is the formula for the area of a trapezoid - half sum of the bases times the altitude - $\frac{(a+b)}{2} \cdot (a+b)$. Looking at the picture another way, this also can be computed as the sum of areas of the three triangles - $\frac{a \cdot b}{2} + \frac{a \cdot b}{2} + \frac{c \cdot c}{2}$. As before, simplifications yield $a^2 + b^2 = c^2$. (There is more to that story.)



Two copies of the same trapezoid can be combined in two ways by attaching them along the slanted side of the trapezoid. One leads to the proof 4, the other to proof 52.

Another proof stems from a rearrangement of rigid pieces, much like proof 2. It makes the algebraic part of proof 4 completely redundant. There is nothing much one can add to the two pictures. (My sincere thanks go to Monty Phister for the kind permission to use the graphics.) There is an interactive simulation to toy with. And another one that clearly shows its relation to proofs 24 or 69.

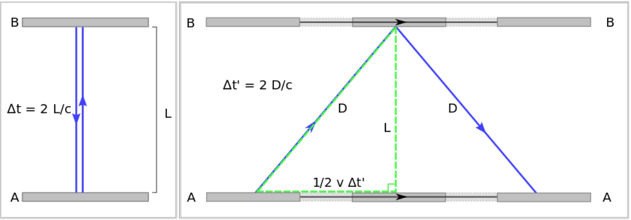


Loomis (pp. 49-50) mentions that the proof "was devised by Maurice Laisnez, a high school boy, in the Junior-Senior High School of South Bend, Ind., and sent to me, May 16, 1939, by his class teacher, Wilson Thornton."

The proof has been published by Rufus Isaac in Mathematics Magazine, Vol. 48 (1975), p. 198.

Time dilation

Simple inference of time dilation due to relative velocity



Left: Observer at rest measures time $2L/c$ between co-local events of light signal generation at A and arrival at A.
Right: Events according to an observer moving to the left of the setup: bottom mirror A when signal is generated at time $t' = 0$, top mirror B when signal gets reflected at time $t' = D/c$, bottom mirror A when signal returns at time $t' = 2D/c$.

Time dilation can be inferred from the observed constancy of the speed of light in all reference frames.
This constancy of the speed of light means, counter to intuition, that speeds of material objects and light are not additive. It is not possible to make the speed of light appear greater by approaching at speed towards the material source that is emitting light. It is not possible to make the speed of light appear less by receding from the source at speed. From one point of view, it is the implications of this unexpected constancy that take away from constancies expected elsewhere.

Consider a simple clock consisting of two mirrors A and B, between which a light pulse is bouncing. The separation of the mirrors is L and the clock ticks once each time the light pulse hits a given mirror.

In the frame where the clock is at rest (diagram on the left), the light pulse traces out a path of length $2L$ and the period of the clock is $2L$ divided by the speed of light $\Delta t = \frac{2L}{c}$.

From the frame of reference of a moving observer traveling at the speed v relative to the rest frame of the clock (diagram at right), the light pulse traces out a longer, angled path. The second postulate of special relativity states that the speed of light in free space is constant for all inertial observers, which implies a lengthening of the period of this clock from the moving observer's perspective.

That is to say, in a frame moving relative to the clock, the clock appears to be running more slowly. Straightforward application of the Pythagorean theorem leads to the well-known prediction of special relativity:

The total time for the light pulse to trace its path is given by $\Delta t' = \dfrac{2D}{c}$.

If $t = t' = 0$, we have $x = vt'$, $2L = ct$ and $2D = ct'$. Next $x/(2D) = vt'/(ct') = v/c := \omega$. Hence

$$\frac{t}{t'} = \frac{L}{D} = \sin\alpha = \sqrt{1-\omega^2} := \gamma^{-1}.$$

We also can prove it using only Pythagora's theorem.
The length of the half path can be calculated as a function of known quantities as $D = \sqrt{\left(\frac{1}{2} v \Delta t'\right)^2 + L^2}$.

Substituting D from this equation into the previous and solving for $\Delta t'$ gives:

$$\Delta t' = \frac{1}{c}\sqrt{(v\Delta t')^2 + (2L)^2}$$

$$(\Delta t')^2 = \frac{v^2}{c^2}(\Delta t')^2 + \left(\frac{2L}{c}\right)^2$$

$$\left(1 - \frac{v^2}{c^2}\right)(\Delta t')^2 = \left(\frac{2L}{c}\right)^2$$

$$(\Delta t')^2 = \frac{(2L/c)^2}{1 - v^2/c^2}$$

$$\Delta t' = \frac{2L/c}{\sqrt{1 - v^2/c^2}}$$

and thus, with the definition of Δt:

$$\Delta t' = \frac{\Delta t}{\sqrt{1 - \frac{v^2}{c^2}}}$$

which expresses the fact that for the moving observer the period of the clock is longer than in the frame of the clock itself.

Thank you for attention!

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