Applications of Abel and Tauber theorems for the wavelet transformation

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1. Abelian type results and applications
2. Tauberian type results and applications

Multidimensional Tauberian theorems for wavelet and non-wavelet transforms, Notation

$\mathcal{S}_0(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$ consists of functions with all the moments equal zero.
The space of highly localized function over $\mathbb{H}^{n+1}$, denoted by $\mathcal{S}(\mathbb{H}^{n+1})$, consists of those $\Phi \in C^\infty(\mathbb{H}^{n+1})$ for which

$$\sup_{(x,y)\in\mathbb{H}^{n+1}} \left(y+\frac{1}{y}\right)^{k_1} (1+|x|)^{k_2} \left|\frac{\partial^l}{\partial y^l}\frac{\partial^m}{\partial x^m}\Phi(x,y)\right| < \infty \ ,$$

for all $k_1, k_2, l \in \mathbb{N}$ and $m \in \mathbb{N}^n$.
The canonical topology of this space is defined in the standard way.
$\mathcal{S}(\mathbb{H}^n)$ is a closed subspace of $\mathcal{S}(\mathbb{R}^n)\hat{\otimes}\mathcal{S}[0,\infty)$, where $S[0,\infty)$ is the restriction of elements from $\mathcal{S}(\mathbb{R})$ to the interval $[0,\infty)$. Therefore, $\mathcal{S}(\mathbb{H}^n)$ is a nuclear space.

Multidimensional Tauberian theorems for wavelet and non-wavelet transforms, Notation

Let E be a Banach space. The spaces of E-valued distributions are defined as: $\mathcal{D}'(\mathbb{R}^n, E) = L_b(\mathcal{D}(\mathbb{R}^n), E)$ and $\mathcal{S}'(\mathbb{R}^n, E) = L_b(\mathcal{S}(\mathbb{R}^n), E)$. We will also use the spaces $\mathcal{S}_0'(\mathbb{R}^n, E) = L_b(\mathcal{S}_0(\mathbb{R}^n), E)$ and $\mathcal{S}'(\mathbb{H}^{n+1}, E) = L_b(\mathcal{S}(\mathbb{H}^{n+1}), E)$.
Let $\mathbf{f}$ be in one of these spaces of E-valued generalized functions and let φ be in the corresponding space of test functions; Then $\langle \mathbf{f}, \varphi \rangle = \langle \mathbf{f}(t), \varphi(t) \rangle \in E$. The Fourier transform of $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, E)$ is defined in the usual way, i.e., $\left\langle \hat{\mathbf{f}}(u), \varphi(u) \right\rangle = \langle \mathbf{f}(t), \hat{\varphi}(t) \rangle \ , \quad \varphi \in \mathcal{S}(\mathbb{R}^n)$.

Weak-asymptotic behavior

Slowly varying function

A measurable real valued function, defined and positive on an interval of the form $(0, A]$ (resp. $[A, \infty)$), $A > 0$, is called *slowly varying* at the origin (resp. at infinity) if

$$\lim_{\varepsilon \to 0^+} \frac{L(a\varepsilon)}{L(\varepsilon)} = 1 \quad \left(\text{resp. } \lim_{\lambda \to \infty} \frac{L(a\lambda)}{L(\lambda)} = 1 \right) .$$

Examples $\log^q x + \sin x, \log\log x$. $(\forall \varepsilon > 0, L(x) = o(x^{\varepsilon}))$
L is slowly varying at the origin if and only if there exist measurable functions u and w defined on some interval $(0, B]$, u being bounded and having a finite limit at 0 and w being continuous in $[0, B]$ with $w(0) = 0$, such that

$$L(x) = exp\left(u(x) + \int_x^B \frac{w(t)}{t} dt \right), \quad x \in (0, B].$$

Multidimensional Tauberian theorems for wavelet and non-wavelet transforms, Weak-asymptotics

Boundedness

Let $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, E)$ and let L be slowly varying at the origin (resp. at infinity). We say that $\mathbf{f}$ is weak-asymptotically bounded of degree $\alpha \in \mathbb{R}$ at the point $x_0 \in \mathbb{R}^n$ (resp. at infinity) with respect to L in $\mathcal{S}'(\mathbb{R}^n, E)$ if for each test function $\varphi \in \mathcal{S}(\mathbb{R}^n)$

$$\sup_{\varepsilon \leq 1} \frac{1}{\varepsilon^{\alpha} L(\varepsilon)} \left\| \langle \mathbf{f}\,(x_0 + \varepsilon t)\,, \varphi(t) \rangle \right\| < \infty \tag{1}$$

$$\left(\text{resp. } \sup_{1 \leq \lambda} \frac{1}{\lambda^{\alpha} L(\lambda)} \left\| \langle \mathbf{f}\,(\lambda t)\,, \varphi(t) \rangle \right\| < \infty \right) .$$

$$\mathbf{f}\,(x_0 + \varepsilon t) = O\left(\varepsilon^{\alpha} L(\varepsilon)\right) \quad \text{as } \varepsilon \to 0^{+} \ \text{ in } \mathcal{S}'(\mathbb{R}^n, E) \tag{2}$$

$$\left(\text{resp. } \mathbf{f}\,(\lambda t) = O\left(\lambda^{\alpha} L(\lambda)\right) \quad \text{as } \lambda \to \infty \ \text{ in } \mathcal{S}'(\mathbb{R}^n, E) \right) . \tag{3}$$

Multidimensional Tauberian theorems for wavelet and non-wavelet transforms, Weak-asymptotics

Definition of weak-asymptotics

Let $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, E)$ and let L be slowly varying at the origin (resp. at infinity). We say that $\mathbf{f}$ has weak-asymptotic behavior of degree $\alpha \in \mathbb{R}$ at the point $x_0 \in \mathbb{R}^n$ (resp. at infinity) with respect to L in $\mathcal{S}'(\mathbb{R}^n, E)$ if there exists $\mathbf{g} \in \mathcal{S}'(\mathbb{R}^n, E)$ such that for each test function $\varphi \in \mathcal{S}(\mathbb{R}^n)$ the following limit holds with respect to the norm of E

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^\alpha L(\varepsilon)} \langle \mathbf{f}\,(x_0 + \varepsilon t)\,, \varphi(t) \rangle = \langle \mathbf{g}(t), \varphi(t) \rangle \in E \tag{4}$$

$$\left(\text{resp. } \lim_{\lambda \to \infty} \frac{1}{\lambda^\alpha L(\lambda)} \langle \mathbf{f}\,(\lambda t)\,, \varphi(t) \rangle \right) .$$

Multidimensional Tauberian theorems for wavelet and non-wavelet transforms, weak-asymptotics

In such a case we write,

$$\mathbf{f}\,(x_0+\varepsilon t)\sim\varepsilon^{\alpha}L(\varepsilon)\mathbf{g}(t)\quad\text{as } \varepsilon\to 0^{+}\ \text{ in } \mathcal{S}'(\mathbb{R}^n,E) \tag{5}$$

$$\Big(\text{resp. } \mathbf{f}\,(\lambda t)\sim\lambda^{\alpha}L(\lambda)\mathbf{g}(t)\quad\text{as } \lambda\to\infty\ \text{ in } \mathcal{S}'(\mathbb{R}^n,E)\,\Big)\ . \tag{6}$$

Distributions having weak-asymptotic behavior at infinity are called asymptotically homogeneous generalized functions.

Multidimensional Tauberian theorems for wavelet and non-wavelet transforms, Wavelet and non-vavelet transforms

Avareaging transform

Let $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, E)$.

$$M^{\mathbf{f}}_{\varphi}(x,y) := (\mathbf{f} * \varphi_y)(x) \in E \ , \quad (x,y) \in \mathbb{H}^{n+1} \ , \tag{7}$$

the standard average of $\mathbf{f}$ with respect to to the test function $\varphi \in \mathcal{S}(\mathbb{R}^n)$.
Notice that $M^{\mathbf{f}}_{\varphi} \in C^{\infty}(\mathbb{H}^{n+1}, E)$.
Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be so that $\mu_0(\phi) = \int_{\mathbb{R}^n} \phi(t)\mathrm{d}t = 1$. *The* $\phi-$*transform* of $\mathbf{f}$ is

$$F_{\phi}\mathbf{f}(x,y) := M^{\mathbf{f}}_{\check{\phi}}(x,y) = \langle \mathbf{f}(x+yt), \phi(t) \rangle \in E \ , \quad (x,y) \in \mathbb{H}^{n+1} \ . \tag{8}$$

Multidimensional Tauberian theorems for wavelet and non-wavelet transforms, Averageing transforms

Wavelet transform

Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be so that $\mu_0(\psi) = \int_{\mathbb{R}^n} \psi(t)\mathrm{d}t = 0$, we then call ψ a *wavelet*. The wavelet transform of $\mathbf{f}$ with respect to ψ is defined by

$$\mathcal{W}_\psi \mathbf{f}(x,y) := M^{\mathbf{f}}_{\check{\bar{\psi}}}(x,y) = \left\langle \mathbf{f}(x+yt), \bar{\psi}(t) \right\rangle \in E \ , \quad (x,y) \in \mathbb{H}^{n+1} \ . \quad (9)$$

Take ψ given in the Fourier side by $\hat{\psi}(u) = e^{-|u| - \frac{1}{|u|}}$, $u \in \mathbb{R}^n$, it satisfies all the requirements.

Let $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, E)$ and let $\phi \in \mathcal{S}(\mathbb{R}^n)$, then

$$\lim_{y \to 0^+} M^{\mathbf{f}}_{\varphi}(\cdot, y) = \mu_0(\varphi)\mathbf{f} \ , \quad \text{in } \mathcal{S}'(\mathbb{R}^n, E) \ .$$

Multidimensional Tauberian theorems for wavelet and non-wavelet transforms, Averageing transforms

Synthesis operator

Let $\psi \in \mathcal{S}_0(\mathbb{R}^n)$. We have that $\mathcal{W}_\psi : \mathcal{S}_0(\mathbb{R}^n) \mapsto \mathcal{S}(\mathbb{H}^{n+1})$ is a continuous linear map . Let $\Phi \in \mathcal{S}(\mathbb{H}^{n+1})$ The *wavelet synthesis operator* with respect to the wavelet ψ is

$$\mathcal{M}_\psi \Phi(t) = \int_0^\infty \int_{\mathbb{R}^n} \Phi(x, y) \frac{1}{y^n} \psi\left(\frac{t-x}{y}\right) \frac{\mathrm{d}x\mathrm{d}y}{y} , \quad t \in \mathbb{R}^n . \tag{10}$$

$\mathcal{M}_\psi : \mathcal{S}(\mathbb{H}^{n+1}) \to \mathcal{S}_0(\mathbb{R}^n)$ is continuous

Multidimensional Tauberian theorems for wavelet and non-wavelet transforms, Averageing transforms

Remark

One can consider $(\mathcal{S}_0)^{\alpha}_{\beta}$ and the corresponding space $\mathcal{S}(\mathbb{H}^{n+1})^{\alpha'}_{\beta'}$ and try to develope the whole theory in this case....

Wavelet Analysis on $\mathcal{S}_0'(\mathbb{R}^n, E)$

Proposition

If ψ admits the reconstruction wavelet η, one has the reconstruction formula for the wavelet transform on $\mathcal{S}_0(\mathbb{R}^n)$

$$\mathrm{Id}_{\mathcal{S}_0(\mathbb{R}^n)} = \frac{1}{c_{\psi,\eta}} \mathcal{M}_\eta \mathcal{W}_\psi \ . \tag{11}$$

(11)is extended to $\mathcal{S}_0'(\mathbb{R}^n)$ via duality arguments

$$\int_0^\infty \int_{\mathbb{R}^n} \mathcal{W}_\psi f(x,y) \Phi(x,y) \frac{\mathrm{d}x\mathrm{d}y}{y} = \left\langle f(t), \mathcal{M}_{\bar{\psi}} \Phi\,(t) \right\rangle \ ,$$

valid for $\Phi \in \mathcal{S}(\mathbb{H}^{n+1})$ and $f \in \mathcal{S}_0'(\mathbb{R}^n)$.

Wavelet Analysis on $\mathcal{S}_0'(\mathbb{R}^n, E)$

Theorem

Let $\psi \in \mathcal{S}_0(\mathbb{R}^n)$ be non-degenerate and let $\eta \in \mathcal{S}_0(\mathbb{R}^n)$ be a reconstruction wavelet for it. Then, we have the inversion formula

$$\mathrm{Id}_{\mathcal{S}_0'(\mathbb{R}^n,E)} = \frac{1}{c_{\psi,\eta}} \mathcal{M}_\eta \mathcal{W}_\psi \ . \tag{12}$$

Furthermore, we have the desingularization formula,

$$\langle \mathbf{f}(t), \rho(t) \rangle = \frac{1}{c_{\psi,\eta}} \int_0^\infty \int_{\mathbb{R}^n} \mathcal{W}_\psi \mathbf{f}(x,y) \mathcal{W}_{\bar{\eta}} \rho(x,y) \frac{\mathrm{d}x\mathrm{d}y}{y} \ , \tag{13}$$

for all $\mathbf{f} \in \mathcal{S}_0'(\mathbb{R}^n, E)$ and $\rho \in \mathcal{S}_0(\mathbb{R}^n)$.

Abelian theorem

Let L be slowly varying at the origin (resp. at infinity) and let $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, E)$ be weak-asymptotically bounded of degree α at the point x_0 (resp. at infinity) with respect to L in $\mathcal{S}'(\mathbb{R}^n, E)$. Then, there exist $k, l \in \mathbb{N}$, $C > 0$ and $\varepsilon_0 > 0$ (resp. $\lambda_0 > 1$) such that for all $(x, y) \in \mathbb{H}^{n+1}$

$$\left\| M_{\varphi}^{\mathbf{f}}(x_0 + \varepsilon x, \varepsilon y) \right\| \leq C\varepsilon^{\alpha} L(\varepsilon) \left(\frac{1}{y} + y \right)^k (1 + |x|)^l \ , \quad 0 < \varepsilon \leq \varepsilon_0 \tag{14}$$

$$\left(\text{resp. } \left\| M_{\varphi}^{\mathbf{f}}(\lambda x, \lambda y) \right\| \leq C\lambda^{\alpha} L(\lambda) \left(\frac{1}{y} + y \right)^k (1 + |x|)^l \ , \quad \lambda_0 \leq \lambda \right). \tag{15}$$

Abelian theorem

Let

$$C_{x_0,\vartheta} = \left\{(x,y) \in \mathbb{H}^{n+1} : |x - x_0| \leq (\tan\vartheta) y\right\} \ \vartheta/geq0.$$

So, $C_{x_0,\vartheta} = (x_0, 0) + C_{0,\vartheta}$. Let $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, E)$

$$\mathbf{f}\,(x_0 + \varepsilon t) \sim \varepsilon^{\alpha} L(\varepsilon)\mathbf{g}(t) \quad \text{as } \varepsilon \to 0^+ \quad \text{in } \mathcal{S}'(\mathbb{R}^n, E)$$

$$\left(\text{resp. } \mathbf{f}\,(\lambda t) \sim \lambda^{\alpha} L(\lambda)\mathbf{g}(t) \quad \text{as } \lambda \to \infty \quad \text{in } \mathcal{S}'(\mathbb{R}^n, E)\,\right) \ .$$

Then, if $0 \leq \vartheta < \pi/2$,

$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in C_{0,\vartheta}}} |(x,y)|^{-\alpha} \left\| \frac{1}{L\left(|(x,y)|\right)} M^{\mathbf{f}}_{\varphi}(x_0 + x, y) - M^{\mathbf{g}}_{\varphi}(x,y) \right\| = 0 \quad (16)$$

Continuation

$$\left(\text{resp.} \lim_{\substack{|(x,y)|\to\infty \\ (x,y)\in C_{0,\vartheta}}} |(x,y)|^{-\alpha} \left\| \frac{1}{L\left(|(x,y)|\right)} M_\varphi^{\mathbf{f}}(x_0+x,y) - M_\varphi^{\mathbf{g}}(x,y) \right\| = 0 \right);$$

in particular, for each fixed point $(x,y) \in \mathbb{H}^{n+1}$,

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^\alpha L(\varepsilon)} M_\varphi^{\mathbf{f}}(x_0+\varepsilon x, \varepsilon y) = M_\varphi^{\mathbf{g}}(x,y) \quad \text{in } E \tag{17}$$

$$\left(\text{resp.} \lim_{\lambda\to\infty} \frac{1}{\lambda^\alpha L(\lambda)} M_\varphi^{\mathbf{f}}(\lambda x, \lambda y) = M_\varphi^{\mathbf{g}}(x,y)\right). \tag{18}$$

Furthermore, an estimate of the form (14) (resp. (15)) holds.

Example

Point values of distributions

An important special case of weak-asymptotic behavior is the value of distributions at a point in the sense of Łojasiewicz, which is obtained when $\alpha = 0$ and $L = 1$. A distribution f is said to have a (distributional) point value at x_0 in the sense of Łojasiewicz if

$$\lim_{\varepsilon \to 0} f(x_0 + \varepsilon x) = \gamma \quad \text{in } \mathcal{D}'(\mathbb{R}).$$

Example

Construction of distributions with no point values in the sense of Łojasiewicz

Let $\{\lambda_n\}_{n=0}^{\infty}$ be a lacunary sequence with $\lambda_{n+1}/\lambda_n > \sigma > 1$. Let $f \in \mathcal{S}'(\mathbb{R})$ have a series representation $f(x) = \sum_{n=0}^{\infty} c_n e^{i\lambda_n x}$, where the series is convergent in $\mathcal{S}'(\mathbb{R})$. Furthermore, suppose that at a given x_0 the point value $f(x_0)$ exists in the sense of Łojasiewicz. Then, by selecting $\psi \in \mathcal{S}_0(\mathbb{R})$ with supp $\overline{\hat{\psi}} \subset [\sigma^{-\frac{1}{2}}, \sigma^{\frac{1}{2}}]$ and $\overline{\hat{\psi}}(1) = 1$, the lacunarity of $\{\lambda_n\}_{n\in\mathbb{N}_0}$ implies

$$\mathcal{W}_\psi f(x_0, \lambda_m^{-1}) = \sum_{n=0}^{\infty} c_n e^{i\lambda_n x_0} \overline{\hat{\psi}}\left(\frac{\lambda_n}{\lambda_m}\right) = c_m e^{i\lambda_m x_0},$$

Example

So, the existence of the distributional point value $f(x_0)$ and Abelian theorem (with $\alpha = 0, L = 1, \mathcal{W}_\psi 1(0,1) = 0$) imply that $c_m e^{i\lambda_m x_0} = o(1)$, or,

$$\lim_{m\to\infty} c_m = 0. \tag{19}$$

Therefore, (19) is a necessary condition for the existence of the distributional point value of f at x_0. On the other hand, we have just shown: *If (19) is violated, then f cannot have distributional point values anywhere.* The same argument we can apply to distributions of the form $\sum_{n=0}^{\infty} c_n \cos(\lambda_n x)$ and $\sum_{n=0}^{\infty} c_n \sin(\lambda_n x)$.

Example

A stronger conclusion than the usual nowhere differentiability for Weierstrass's function

We observe the Weierstrass function

$$w(x) = \sum_{n=0}^{\infty} \gamma^{-n} \cos(\beta^n x), \quad \beta \geq \gamma > 1$$

and its first derivative

$$w'(x) = -\sum_{n=0}^{\infty} \left(\frac{\beta}{\gamma}\right)^n \sin(\beta^n x).$$

Since $(\beta/\gamma)^n \neq o(1)$, it follows from previous Example that $w'(x_0)$ does not exist in the sense of Łojasiewicz at any $x_0 \in \mathbb{R}$. In particular, w is nowhere differentiable.

Wavelet Tauberian Characterization of Quasiasymptotics in $\mathcal{S}_0'(\mathbb{R}^n, E)$

Proposition

Let $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, E)$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$, and let L be slowly varying at the origin (resp. at infinity). Then,

$$\limsup_{\varepsilon \to 0^+} \sup_{|x|^2+y^2=1,\, y>0} \frac{y^k}{\varepsilon^\alpha L(\varepsilon)} \left\| M_\varphi^{\mathbf{f}}(x_0 + \varepsilon x, \varepsilon y) \right\| < \infty \tag{20}$$

$$\left(\text{resp. } \limsup_{\lambda \to \infty} \sup_{|x|^2+y^2=1,\, y>0} \frac{y^k}{\lambda^\alpha L(\lambda)} \left\| M_\varphi^{\mathbf{f}}(\lambda x, \lambda y) \right\| < \infty \right) . \tag{21}$$

implies an estimate of the form (14) (resp. (15)).

Wavelet Tauberian Characterization of weak-asymptotics in $\mathcal{S}_0'(\mathbb{R}^n, E)$

Proposition

Let $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, E)$, $\psi \in \mathcal{S}_0(\mathbb{R}^n)$, and let L be slowly varying at the origin (resp. at infinity). Assume that the wavelet ψ is non-degenerate. Then, the existence of the limits

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^\alpha L(\varepsilon)} \mathcal{W}_\psi \mathbf{f}(x_0 + \varepsilon x, \varepsilon y) = W_{x,y} \ , \quad \text{for each } (x,y) \in \mathbb{H}^{n+1} \cap \mathbb{S}^n \ , \tag{22}$$

$$\left(\text{resp. } \lim_{\lambda\to\infty} \frac{1}{\lambda^\alpha L(\lambda)} \mathcal{W}_\psi \mathbf{f}(\lambda x, \lambda y) = W_{x,y} \in E\right) \tag{23}$$

and the existence of $k \in \mathbb{N}$ such that

Wavelet Tauberian Characterization of weak-asymptotics in $\mathcal{S}_0'(\mathbb{R}^n, E)$

$$\limsup_{\varepsilon\to 0^+} \sup_{|x|^2+y^2=1,\, y>0} \frac{y^k}{\varepsilon^\alpha L(\varepsilon)} \left\| \mathcal{W}_\psi \mathbf{f}\,(x_0+\varepsilon x, \varepsilon y)\right\| < \infty \tag{24}$$

$$\left(\text{resp. } \limsup_{\lambda\to \infty} \sup_{|x|^2+y^2=1,\, y>0} \frac{y^k}{\lambda^\alpha L(\lambda)} \left\| \mathcal{W}_\psi \mathbf{f}\,(\lambda x, \lambda y)\right\| < \infty\right), \tag{25}$$

are necessary and sufficient for $\mathbf{f}$ to have weak-asymptotic behavior in the space $\mathcal{S}_0'(\mathbb{R}^n, E)$, namely, the existence of an E-valued distribution $\mathbf{g}\in\mathcal{S}'(\mathbb{R}^n, E)$ such that

$$\mathbf{f}(x_0+\varepsilon t)\sim \varepsilon^\alpha L(\varepsilon)\mathbf{g}(t) \quad \text{as } \varepsilon\to 0^+ \ \text{ in } \mathcal{S}_0'(\mathbb{R}^n, E) \tag{26}$$

$$\left(\text{resp. } \mathbf{f}\,(\lambda t)\sim \lambda^\alpha L(\lambda)\mathbf{g}(t) \quad \text{as } \lambda\to\infty \ \text{ in } \mathcal{S}_0'(\mathbb{R}^n, E)\right). \tag{27}$$

In such a case, the restriction of $\mathbf{g}$ to $\mathcal{S}_0(\mathbb{R}^n)$ is uniquely determined by $\mathcal{W}_\psi \mathbf{g}(x,y) = W_{x,y}$.

Theorem

Let $f \in \mathcal{S}'(\mathbb{R}^n, \mathcal{C}^r(K))$, let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be a non-degenerate wavelet, and let L be slowly varying at the origin. Suppose that there exists $k \in \mathbb{N}$ such that

$$\limsup_{\varepsilon \to 0^+} \sup_{|x|^2+y^2=1,\, y>0} \frac{y^k}{\varepsilon^\alpha L(\varepsilon)} \| \mathcal{W}_\psi f\,(t+\varepsilon x, \varepsilon y) \|_{\mathcal{C}^r(K)} < \infty. \tag{28}$$

Condition (28) is necessary and sufficient for:
(i) If $\alpha \notin \mathbb{N}$, there exists a polynomial P_t with values in $\mathcal{C}^r(K)$ such that $f - P$ is weak-asymptotically bounded of degree α with values in $\mathcal{C}^r(K)$ with respect to L in the space $\mathcal{S}'(\mathbb{R}^n, \mathcal{C}^r(K))$.

Tauberian theorems for weak-asymptotic boundedness of vector-valued distribution

(ii) If $\alpha = k \in \mathbb{N}$, there exist a polynomial P_t with values in $\mathcal{C}^r(K)$ and asymptotically homogeneously bounded functions c_m, $|m| = k$, with values in $\mathcal{C}^r(K)$, of degree 0 with respect to L such that f has the following asymptotic expansion

$$f(t + \varepsilon x) = P_t(\varepsilon x) + \varepsilon^k \sum_{|m|=k} x^m c_m(t, \varepsilon) + O\left(\varepsilon^k L(\varepsilon)\right)$$

as $\varepsilon \to 0^+$ in the space $\mathcal{S}'(\mathbb{R}^n, \mathcal{C}^r(K))$.

Characterizations of Hölder functions through the wavelet transform

Proposition

Let $f \in \mathcal{S}'(\mathbb{R})$, $\alpha \in (0,1)$, $I = (A, B)$, and let L be a slowly varying function at 0 which is bounded on compact subsets of $\mathbb{R}_+$. Let $f \in \mathcal{S}'(\mathbb{R})$ be such that

$$|\mathcal{W}_\psi f(b,a)| = o(|a|^\alpha L(a)) \ , \quad \text{as } a \to 0 \ ,$$

uniformly when b remains in compact subsets of I. Then, for every closed subinterval $I' = [A', B'] \subset \mathbb{R}$ there holds

$$|f(t+h) - f(t)| = o(h^\alpha L(h)) \ , \quad \text{as } h \to 0^+ \text{ uniformly in } t \in I' \ .$$

In particular, there exists $M_{I'}$ such that

$$|f(t+h) - f(t)| \le M_I |h|^\alpha L(|h|), \quad t, t+h \in I',$$

Characterizations of Hölder functions through the wavelet transform

Proposition

Let $\alpha \in (0,1)$, $I = (A, B)$, and let L be a slowly varying function at 0 which is bounded on compact subsets of $\mathbb{R}_+$. Let $f \in \mathcal{S}'(\mathbb{R})$ satisfy

$$\sup_{t \in I} |\mathcal{W}_\psi f(t+b, a)| = o(|b|^\alpha L(|b|)), \text{ as } |b| \to 0, \text{ uniformly in } 0 < a < 1 . \tag{29}$$

Then, for any compact subinterval $I' \subset I$, there holds

$$|f(t+h) - f(t)| = o(h^\alpha L(h)) \ , \quad \text{as } h \to 0^+ \text{ uniformly in } t \in I' . \tag{30}$$

In particular, there exists $M_{I'}$ such that

$$|f(t+h) - f(t)| \leq M_I |h|^\alpha L(|h|), \quad t, t+h \in I', \tag{31}$$

where $M_I' > 0$ does not depend on t and h.

Applications to local analysis of distributions

Let $\mathbf{f} \in \mathcal{S}'(\mathbb{R}^n, E)$. Using conditions

$$||\mathbf{f}(\varepsilon x)||_E = O(\varepsilon^s L(\varepsilon)),\ \varepsilon \to 0 \quad \text{and} \tag{32}$$

$$||\mathbf{f}(\lambda x)||_E = O(\lambda^s L(\lambda)),\ \lambda \to \infty, \tag{33}$$

where L is slowly varying at zero and at infinity, we have the next definition related to Meyer spaces:

Definition

$\mathbf{f} \in \dot{\mathcal{O}}_E^{s,L}$, resp, $\mathbf{f} \in \dot{\Gamma}_E^{s,L}$ if and only if conditions (32) and (33) hold in the sense of convergence in $\mathcal{S}'(\mathbb{R}^n, E)$, resp., $\mathcal{S}_0'(\mathbb{R}^n, E)$.
$\mathbf{f} \in \mathcal{O}_E^{s,L}$, resp., $f \in \Gamma_E^{s,L}$ if and only if condition (32) holds in the sense of convergence in $\mathcal{S}'(\mathbb{R}^n, E)$, resp., $\mathcal{S}_0'(\mathbb{R}^n, E)$.

Applications to local analysis of distributions

Let K be a compact set of $\mathbb{R}^n$ ($K \subset\subset \mathbb{R}^n$). The cases when $E = (C(K), ||\cdot||_\infty)$, where $||f||_\infty = \sup_{t\in K}|f(t)|$ and

$$t \mapsto f(t+\cdot) \in L(K, \mathcal{S}(\mathbb{R}^n)) \text{ or } t \mapsto f(t+\cdot) \in L(K, \mathcal{S}_0(\mathbb{R}^n)),$$

are of special interest and we devote a paper to these cases.

Applications to local analysis of distributions

Definition

Let $f \in \mathcal{S}'(\mathbb{R}^n)$. Then

$$f \in O^s(K) \text{ if } \sup_{t \in K} |f(t + \varepsilon x)| = O(\varepsilon^s), \varepsilon \to 0, \text{ in } \mathcal{S}'(\mathbb{R}^n); \tag{34}$$

$$f \in \Gamma^s(K) \text{ if } \sup_{t \in K} |f(t + \varepsilon x)| = O(\varepsilon^s), \varepsilon \to 0, \text{ in } \mathcal{S}'_0(\mathbb{R}^n); \tag{35}$$

We reformulate Boni's definition of $C_0^{s,s'}$ in the general case and in the case when $E = (C(K), || \cdot ||_\infty)$:

Applications to local analysis of distributions

Definition

(i) Let $\mathbf{f} \in \mathcal{S}^{'}(\mathbb{R}^n, E)$. Then $\mathbf{f} \in \mathcal{S}_E^{s,s'}(\mathbb{R}^n)$ if the following conditions hold in $\mathcal{S}^{'}(\mathbb{R}^n, E)$:

$$(a) \ \ ||\mathbf{f} * \varphi(x)||_E \leq C_0(1+|x|)^{-s'}, \ x \in \mathbb{R}^n,$$

$$(b) \ \ ||\mathbf{f} * \psi_\varepsilon(x)||_E \leq C_1 \varepsilon^s (1+|x|/\varepsilon)^{-s'}, \ x \in \mathbb{R}^n, \varepsilon \in (0,1),$$

where $\varphi \in \mathcal{S}(\mathbb{R})^n$ is a mollifier and $\psi \in \mathcal{S}_0(\mathbb{R})^n$ is a wavelet.
(ii) Let $f \in \mathcal{S}'(\mathbb{R}^n)$. Then $f \in C^{s,s'}(K)$ if

$$(a) \ \ \sup_{t \in K} |f * \varphi(t+x)| \leq C_0(1+|x|)^{-s'}, \ x \in \mathbb{R}^n,$$

$$(b) \ \ \sup_{t \in K} |f * \psi_\varepsilon(t+x)| \leq C_1 \varepsilon^s (1+|x|/\varepsilon)^{-s'}, \ x \in \mathbb{R}^n, \varepsilon \in (0,1),$$

Applications to local analysis of distributions

Corollary

Let $s \in (0,1]$, $I = (A,B)$, $K = [A,B]$ and $f \in C^{s,s'}(K)$. Then f is of the Hölder class s on every compact subinterval of I.
The similar conclusion can be derived for elements of $\mathcal{O}^s(I)$ with the condition $s \in (0,1]$. Moreover, analogous results hold in the $n-$dimensional case.

Applications to local analysis of distributions

If $K = \{x_0\}$, $L = 1$, then earlier definition reduces to the definitions of the known spaces $\mathcal{O}^s, \Gamma^s$ and (by translation) $\mathcal{O}^s(x_0), \Gamma^s(x_0)$. Alsoone obtains Boni's definition of $C^{s,s'}_{x_0}$, taking $K = \{x_0\}$.
We note that our Tauberian theorems give another approach to quoted spaces of Mayer and Boni. Moreover, we have more precis extension theorems for distributions defined on $\mathbb{R}^n \setminus \{0\}$ since we have considered also associate asymptoticaly homogeneous functions

Let $\{c_n\}_{n=0}^{\infty}$ be sequence of complex numbers. Recall, $\sum_{n=0}^{\infty} c_n$ is said to be Lambert summable to β if

$$\sum_{n=0}^{\infty} c_n \frac{yn}{e^{yn}-1} \text{ converges for } y>0, \text{ and } \lim_{y\to 0^+} \sum_{n=0}^{\infty} c_n \frac{yn}{e^{yn}-1} = \beta.$$

$$\sum_{n=0}^{\infty} c_n = \beta \quad (\mathrm{L}). \tag{36}$$

Assume that $\{c_n\}_{n=0}^{\infty}$ is of slow growth, i.e., there is $m \in \mathbb{N}$ such that $c_n = O(n^m)$. Then, $f(t) = \sum_{n=0}^{\infty} c_n e^{itn}$ defines a periodic distribution over the real line. Let $\eta \in \mathcal{S}(\mathbb{R})$ be a test function such that $\eta(u) = u/(e^u - 1)$, for $u > 0$. set $\phi(t) = (1/2\pi)\hat{\eta}(t)$; thus, the $\phi-$transform of f is precisely

$$F_\phi f(x,y) = \frac{1}{2\pi}\left\langle e^{ixu}\hat{f}(u), \frac{yu}{e^{yu}-1}\right\rangle = \sum_{n=0}^{\infty} c_n e^{ixn}\frac{yn}{e^{yn}-1}.$$

Set $\phi(t) = (1/2\pi)\hat{\eta}(t)$; thus, the $\phi-$transform of f is precisely

$$F_\phi f(x,y) = \frac{1}{2\pi}\left\langle f(t), \frac{1}{y}\hat{\eta}(\frac{x-t}{y})\right\rangle = \sum_{n=0}^{\infty} c_n e^{ixn}\frac{yn}{e^{yn}-1}.$$

Consequently, (36) is equivalent to an statement on the boundary behavior of the $\phi-$transform

$$\lim_{y\to 0^+} F_\phi f(0,y) = \beta. \tag{37}$$

Laplace transforms as $\phi-$transforms

Let Γ be a closed convex acute cone with vertex at the origin. Its conjugate cone is denoted by Γ^*. Γ^* has non-empty interior; set $C_\Gamma = \operatorname{int}\Gamma^*$ and $T^{C_\Gamma} = \mathbb{R}^n + iC_\Gamma$. We denote by $\mathcal{S}'_\Gamma(E)$ the subspace of E-valued tempered distributions supported by Γ. Given $\mathbf{h} \in \mathcal{S}'_\Gamma(E)$, its *Laplace transform* is

$$\mathcal{L}\left\{\mathbf{h}; z\right\} = \left\langle \mathbf{h}(u), e^{iz\cdot u}\right\rangle, \quad z \in T^{C_\Gamma};$$

It is a holomorphic function on the tube domain T^{C_Γ}. Fix $\omega \in C_\Gamma$. We may write $\mathcal{L}\left\{\mathbf{h}; x + i\sigma\omega\right\}$, $x \in \mathbb{R}^n$, $\sigma > 0$, as a $\phi-$transform. In fact, choose $\eta_\omega \in \mathcal{S}(\mathbb{R}^n)$ such that $\eta_\omega(u) = e^{-\omega\cdot u}$, $u \in \Gamma$; then, with $\phi_\omega = 1/(2\pi)^n \hat{\eta}_\omega$ and $\hat{\mathbf{f}} = (2\pi)^n \mathbf{h}$,

$$\mathcal{L}\left\{\mathbf{h}; x + i\sigma\omega\right\} = F_{\phi_\omega}\mathbf{f}(x,\sigma). \tag{38}$$

$$\left\langle \mathbf{h}(u), e^{iz\cdot u}\right\rangle = \left\langle \mathbf{h}(u), e^{i(x+i\sigma\omega)\cdot u}\right\rangle$$

$$= \left\langle \mathbf{f}(t), (2\pi)^{-n}\frac{1}{\sigma^n}\hat{\eta}_\omega(\frac{t-x}{\sigma})\right\rangle = F_{\phi_\omega}\mathbf{f}(x,\sigma)$$

Littlewood's Tauberian theorem. The classical Tauberian theorem of Littlewood states that if

$$\lim_{\varepsilon\to 0^+}\sum_{n=0}^{\infty} c_n e^{-\varepsilon n}=\beta \tag{39}$$

and if the Tauberian hypothesis $c_n = O(1/n)$ is satisfied, then the numerical series is convergent, i.e., $\sum_{n=0}^{\infty} c_n = \beta$.

We give a quick proof of this theorem.

I $h(u) = \sum_{n=0}^{\infty} c_n \delta(u-n)$ has the quasiasymptotic behavior

$$h(\lambda u) = \sum_{n=0}^{\infty} c_n \delta(\lambda u - n) \sim \beta \frac{\delta(u)}{\lambda} \quad \text{as } \lambda \to \infty \text{ in } \mathcal{S}'(\mathbb{R}). \tag{40}$$

II The convergence of the series can be established from (40).

Pointwise Analysis of Riemann Type Distributions at the Rationals

We investigate the pointwise weak-asymptotic expansion of the family of *Riemann distributions*

$$R_\beta(t) = \sum_{n=1}^{\infty} \frac{e^{i\pi n^2 t}}{n^{2\beta}} \in \mathcal{S}'(\mathbb{R}_t), \quad \beta \in \mathbb{C},$$

at points of $\mathbb{Q}$. We split $\mathbb{Q}$ into two disjoint subsets S_0 and S_1 where

$$S_0 = \left\{ \frac{2\nu+1}{2j} : \nu, j \in \mathbb{Z} \right\} \cup \left\{ \frac{2j}{2\nu+1} : \nu, j \in \mathbb{Z} \right\}$$

and

$$S_1 = \left\{ \frac{2\nu+1}{2j+1} : \nu, j \in \mathbb{Z} \right\}.$$

Pointwise Analysis of Riemann Type Distributions at the Rationals

When $\beta > 1/2$, R_β is a continuous function. The imaginary part of R_1 is the classical Riemann "non-differentiable" function. It is well known that if $\beta > 3/4$, then R_β is differentiable at the points of S_1 and has local cusps with differentiable remainder at points of S_0; for $\beta \in (1/2, 5/4)$, R_β is not differentiable at any irrational point, as shown essentially by Hardy and Littlewood . Jaffard and Meyer showed that $\Im m R_1$ has trigonometric chirps at the points of S_1. We will exhibit more precise information concerning the scaling weak-asymptotic properties of R_β at the rationals, in fact, we will show that R_β admits a full weak-asymptotic series at points of $\mathbb{Q}$, no matter the value of β.

Pointwise Analysis of Riemann Type Distributions at the Rationals

In particular, our analysis reveals that R_β has weak scaling exponent equal to ∞ at points of S_1; at points of S_0, it has infinite weak scaling exponent after subtraction of an adequate term. To this end, we will be led to the study of the analytic continuation of the zeta-type function

$$\zeta_r(z) := \sum_{n=1}^{\infty} \frac{e^{i\pi r n^2}}{n^z}, \quad \Re e\, z > 1, \tag{41}$$

where $r \in \mathbb{Q}$. If $r = 0$, (41) reduces to $\zeta_0 = \zeta$, the familiar Riemann zeta function.

Pointwise Analysis of Riemann Type Distributions at the Rationals

We now determine the weak-asymptotic expansion of R_0 at 1. Observe that $R_0(1+t) = 2R_0(4t) - R_0(t)$, thus the behavior at origin implies that

$$R_0(1+\varepsilon t) = -\frac{1}{2} + o(\varepsilon^{\infty}) \quad \text{as } \varepsilon \to 0^{+} \text{ in } \mathcal{S}'(\mathbb{R}_t). \tag{42}$$

We return to the general case. Consider the two complex transformations

$$Kz = z+1 \quad \text{and} \quad Uz = -1/z, \quad \text{for } z \in \mathbb{C},$$

they generate the well know modular group which leaves invariant the upper half-plane and the real line. We are more interested in the *theta group*, namely, the subgroup G_{ϑ} of modular transformations generated by K^2 and U.

Pointwise Analysis of Riemann Type Distributions at the Rationals

Then, one readily verifies that

$$G_\vartheta \cdot 0 = S_0 \quad \text{and} \quad G_\vartheta \cdot 1 = S_1,$$

that is, S_0 is the orbit of 0 under G_θ while S_1 that of 1. Let ϑ be the *Jacobi theta* function given by

$$\vartheta(z) := 1 + 2\sum_{n=1}^{\infty} e^{i\pi n^2 z}, \quad \Im m\, z > 0.$$

We then have the following transformation laws

$$\vartheta(K^2 z) = \vartheta(z) \quad \text{and} \quad \vartheta(Uz) = \sqrt{-iz}\,\vartheta(z);$$

the first of them is completely obvious, while the second one follows easily from the Poisson summation formula. Observe that ϑ admits a boundary tempered distribution on the real line, which we also denote by ϑ, or $\vartheta(t)$.

Pointwise Analysis of Riemann Type Distributions at the Rationals

The ensuing lemma describes the scaling weak-asymptotic properties of R_0 at points of the orbit $S_1 = G_\vartheta \cdot 1$.

Lemma

Let $r \in G_\vartheta \cdot 1$. Then, $R_0 \in C_w^\infty(r)$. Furthermore, at those points, $R_0(r) = -1/2$ and $R_0^{(m)}(r) = 0$, distributionally, for each $m \geq 1$.

We may rephrase Lemma 1 by saying that $R_0 + 1/2 \in \mathcal{O}^\infty(r)$ for each $r \in G_\vartheta \cdot 1$.

Observe that Lemma 1 gives then the full weak-asymptotic expansion of R_0 at $r = (2j+1)/(2\nu+1)$, $j, \nu \in \mathbb{Z}$, namely,

$$R_0\left(r + \varepsilon t\right) = -\frac{1}{2} + o(\varepsilon^\infty) \quad \text{as } \varepsilon \to 0^+ \text{ in } \mathcal{S}'(\mathbb{R}_t).$$

Pointwise Analysis of Riemann Type Distributions at the Rationals

Theorem

Let $r \in G_\theta \cdot 1$*. Then* $R_\beta \in C^\infty_w(r)$ *for any* $\beta \in \mathbb{C}$*. Moreover, the Dirichlet series*

$$\zeta_r(z) = \sum_{n=1}^{\infty} \frac{e^{i\pi r n^2}}{n^z} \quad (\mathrm{C}), \quad z \in \mathbb{C}, \tag{43}$$

defines an entire function in z*, where the sums of series for* $\Re e\, z < 1$ *are taken in the Cesàro sense, and they are convergent on the closed half-plane* $\Re e\, z \geq 1$*. In particular, the Łojasiewicz point values of the derivatives of* R_β *at points of the orbit* $G_\theta \cdot 1$ *are given by*

$$R_\beta^{(m)}(r) = (i\pi)^m \zeta_r(2\beta - 2m), \quad \textit{distributionally}, \quad \textit{for all } m \in \mathbb{N}. \tag{44}$$

Pointwise Analysis of Riemann Type Distributions at the Rationals

It is implicit in Theorem 2 that R_β admits a weak-asymptotic series at the points of $G_\theta \cdot 1$.

Corollary

Let $r \in G_\theta \cdot 1$. Then, for any $\beta \in \mathbb{C}$,

$$R_\beta\left(r + \varepsilon t\right) \sim \sum_{m=0}^{\infty} \frac{\zeta_r(2\beta - 2m)}{m!} (i\varepsilon\pi t)^m \quad \text{as } \varepsilon \to 0^+ \text{ in } \mathcal{S}'(\mathbb{R}_t).$$

Pointwise Analysis of Riemann Type Distributions at the Rationals

We now proceed to study the pointwise properties of R_β on the orbit $G_\theta \cdot 0$. As usual, we start with R_0.

Theorem

At any point $r \in G_\vartheta \cdot 0$, there exists a constant $\mathfrak{p}_r \in \mathbb{C}$ such that

$$R_0(t) - \frac{\sqrt{i}}{2}\mathfrak{p}_r(t - r + i0)^{-\frac{1}{2}} + 1/2 \in \mathcal{O}^\infty(r).$$

Moreover, the constants $\mathfrak{p}_r$ are completely determined by the transformation equations:

$$\mathfrak{p}_0 = 1, \quad \mathfrak{p}_{K^2 r} = \mathfrak{p}_r, \quad \textit{and} \quad \mathfrak{p}_{Ur} = \sqrt{-\frac{i}{r}}\,\mathfrak{p}_r. \tag{45}$$

Theorem 4 means that, at any point of the orbit $G_\vartheta \cdot 0$, we have the weak-asymptotic expansion

$$R_0(r+\varepsilon t) = \frac{\sqrt{i}}{2}\mathfrak{p}_r \varepsilon^{-\frac{1}{2}}(t+i0)^{-\frac{1}{2}} - \frac{1}{2} + o(\varepsilon^\infty) \quad \text{as } \varepsilon \to 0^+ \text{ in } \mathcal{S}'(\mathbb{R}_t). \quad (46)$$

Pointwise Analysis of Riemann Type Distributions at the Rationals

Depending on whether $\beta = 1/2$ or $\beta \neq 1/2$, the distributions R_β will behave differently on the orbit of 0 under the theta group. This fact is intimately connected with the analytic continuation of ζ_r for $r \in G_\theta \cdot 0$, which is obtained in the next theorem.

Theorem

Let $r \in G_\vartheta \cdot 0$. Then, ζ_r admits an analytic continuation to $\mathbb{C} \setminus \{1\}$. Furthermore, ζ_r has a simple pole at $z = 1$ with residue $\mathfrak{p}_r$, determined by (45), and the entire function

$$A_r(z) = \zeta_r(z) - \frac{\mathfrak{p}_r}{z-1} \tag{47}$$

can be expressed as the Cesàro limit

$$A_r(z) = \lim_{x \to \infty} \sum \frac{e^{i\pi r n^2}}{n^z} - \mathfrak{p}_r \int_1^x \frac{\mathrm{d}\xi}{\xi^z} \quad (\mathrm{C}). \tag{48}$$

Pointwise Analysis of Riemann Type Distributions at the Rationals

We describe the pointwise behavior of R_β on $G_\vartheta \cdot 0$. We define the *generalized gamma constant* as $\gamma_r := A_r(1)$. Observe that in fact $\gamma_0 = \gamma$, the familiar Euler gamma constant because $\zeta_0 = \zeta$ is the Riemann zeta function.

Pointwise Analysis of Riemann Type Distributions at the Rationals

Theorem

Let $r \in G_\vartheta \cdot 0$.

(i) *If* $\beta \in \mathbb{C} \setminus \{1/2\}$, *then*

$$R_\beta(r+\varepsilon t) \sim \frac{(-i\pi)^{\beta-\frac{1}{2}}\Gamma\left(\frac{1}{2}-\beta\right)\mathfrak{p}_r}{2}(\varepsilon t+i0)^{\beta-\frac{1}{2}}+\sum_{m=0}^{\infty}\frac{\zeta_r(2\beta-2m)}{m!}(i\varepsilon\pi t$$

as $\varepsilon \to 0^+$ *in* $\mathcal{S}'(\mathbb{R}_t)$.

(ii) *When* $\beta = 1/2$, *we have*

$$R_{\frac{1}{2}}(r+\varepsilon t) \sim \gamma_r+\frac{\mathfrak{p}_r}{2}\left(-\log\left(\frac{\varepsilon\,|t|}{\pi}\right)+\frac{i\pi}{2}\operatorname{sgn} t-\gamma\right)+\sum_{m=1}^{\infty}\frac{\zeta_r(1-2m)}{m!}(i\varepsilon$$

as $\varepsilon \to 0^+$ *in* $\mathcal{S}'(\mathbb{R}_t)$.

Pointwise Analysis of Riemann Type Distributions at the Rationals

We discuss some useful formulas which can be derived from our previous analysis. The next corollary provides formulas for the constants $\mathfrak{p}_r$.

Corollary

Let $r \in G_\vartheta \cdot 0$. Then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{i\pi r n^2} = \mathfrak{p}_r. \tag{49}$$

Pointwise Analysis of Riemann Type Distributions at the Rationals

We now give a formula for γ_r.

Corollary

Let $r \in G_\vartheta \cdot 0$. The series

$$\sum_{n=0}^{\infty} \frac{e^{i\pi r n^2} - \mathfrak{p}_r}{n^{1+iy}}\,, \tag{50}$$

is convergent for any $y \in \mathbb{R}$. In particular,

$$\sum_{n=0}^{\infty} \frac{e^{i\pi r n^2} - \mathfrak{p}_r}{n} = \gamma_r - \mathfrak{p}_r \gamma \tag{51}$$

or equivalently,

$$\lim_{x\to\infty} \sum_{n=1}^{N} \frac{e^{i\pi r n^2}}{n} - \mathfrak{p}_r \log N = \gamma_r. \tag{52}$$

We state it in the following corollary.

Corollary

The series

$$\sum_{n=1}^{\infty} \frac{e^{i\pi n^2 \frac{2j+1}{2\nu+1}}}{n^{1+yi}}$$

is convergent for any $j, \nu \in \mathbb{Z}$ *and* $y \in \mathbb{R}$.

The pointwise behavior of R_0 can be used to calculate some Cesàro sums and limits which apparently have not been given elsewhere before. That is the context of the next corollary, whose proof is obtained immediately by comparing Corollary 3 with Lemma 1 and the expansion from Theorem 6 with (46).

Pointwise Analysis of Riemann Type Distributions at the Rationals

Corollary

For any $j, \nu \in \mathbb{Z}$*,*

$$\sum_{n=1}^{\infty} n^{2m} e^{i\pi n^2 \frac{2j+1}{2\nu+1}} = 0 \quad (\mathrm{C}), \qquad m = 1, 2, 3, \ldots,$$

and

$$\sum_{n=1}^{\infty} e^{i\pi n^2 \frac{2j+1}{2\nu+1}} = -\frac{1}{2} \quad (\mathrm{C}).$$

If $r \in G_{\vartheta} \cdot 0,$ *then*

$$\lim_{x\to\infty} \left(\sum_{1 \le n < x} n^{2m} e^{i\pi r n^2} - \mathfrak{p}_r \int_0^x \xi^{2m} \mathrm{d}\xi \right) = 0 \quad (\mathrm{C}), \qquad m = 1, 2, 3, \ldots,$$