On the class $\mathcal{U}$ of univalent functions

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Organization

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Notations and preliminaries

- $\mathcal{A} = \{f : f \text{ is analytic in } \mathbb{D}, f(0) = 0, f'(0) = 1\}$, $\mathbb{D} = \{z : |z| < 1\}$
- $\mathcal{S} = \{f \in \mathcal{A} : f \text{ is univalent in } \mathbb{D}\}$
- $\mathcal{U} = \left\{ f \in \mathcal{A} : \left| f'(z) \left(\frac{z}{f(z)} \right)^2 - 1 \right| < 1 \right\}$
- $\mathcal{S}^* = \{f \in \mathcal{S} : f(\mathbb{D}) \text{ is starlike } \}$

(Aksentiev, 1958; Ozaki-Nunokawa, 1972;Ob-Po,2006)

- $\mathcal{U} \subsetneq \mathcal{S}$
- $k(z) = \frac{z}{(1-z)^2} \in \mathcal{U}$
- $f \in \mathcal{S} \Longrightarrow r^{-1}f(rz) \in \mathcal{U}$ for $0 < r \leq \frac{1}{\sqrt{2}}$.

  The result is sharp as the function $f(z) = \frac{z(1-\frac{1}{\sqrt{2}}z)}{1-z^2}$ shows.

Theorem 1(Ob-Po, 2009)

- $\mathcal{U} \not\subset \mathcal{S}^*$, and $f(z) = \dfrac{z}{1+\frac{1}{2}z+\frac{1}{2}z^3} \in \mathcal{U} \setminus \mathcal{S}^*$.

The class $\mathcal{U}$ contains the next subclasses:

1. $(f \in \mathcal{A}\ , f(z) = z + a_2z^2 + \dots$ and $\sum_{n=1}^{\infty} n|a_n| \leq 1) \Longrightarrow f \in \mathcal{U}$
2. $(f \in \mathcal{S}^{\star}$ *and* $f(z) = z - a_2z^2 - a_3z^3 - \dots$, with $a_n \geq 0)$ $\Longrightarrow f \in \mathcal{U}$ (Silverman, 1975)
3. $(f \in \mathcal{A}$ *and* $|(\frac{z}{f(z)})''| \leq 2) \Longrightarrow f \in \mathcal{U}$

Theorem 2(Prawitz, 1926)

If $f \in \mathcal{S}$ and $\frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n$, then $\displaystyle\sum_{n=2}^{\infty}(n-1)|b_n|^2 \leq 1$

Theorem 3(Barnard, Naik,Ob.,Po., 2004)

Let $f, g \in \mathcal{S}$ with the representations
$\frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n$, $\quad \frac{z}{g(z)} = 1 + \sum_{n=1}^{\infty} c_n z^n$. If
$\Phi(z) = \frac{z}{f(z)} * \frac{z}{g(z)} = 1 + \sum_{n=1}^{\infty} b_n c_n z^n \neq 0$ for every $z \in \mathbb{D}$, then
$F(z) = \dfrac{z}{\Phi(z)} \in \mathcal{U}$ and, in particular, F is univalent in $\mathbb{D}$.

Theorem 4[Ob-Po, 2001]

Let $f \in \mathcal{A}$ and $\frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n$, then

(i) $\sum_{n=2}^{\infty}(n-1)|b_n| \leq 1 \Longrightarrow f \in \mathcal{U}$. This condition is not necessary.

Example. For the function f defined by $\frac{z}{f(z)} = 1 + \frac{1}{3}z^2 + \frac{\sqrt{5}}{6}iz^3 + \frac{1}{9}z^4$ we have that $f \in \mathcal{U}$, but $\sum_{n=2}^{\infty}(n-1)|b_n| = \frac{1}{3} + \frac{\sqrt{5}}{3} + \frac{1}{3} > 1$

(ii) $$f \in \mathcal{U} \Longrightarrow \sum_{n=2}^{\infty}(n-1)^2|b_n|^2 \leq 1 \Longrightarrow |b_n| \leq \frac{1}{n-1}$$

$(|b_1| \leq 2)$. Sharp for f such that $\dfrac{z}{f(z)} = 1 + \dfrac{z^n}{n-1}, \ n \geq 2.$

Main Results

Theorem 5(Ob-Po, 2009)

Let $f \in \mathcal{A}$ with $\frac{z}{f(z)} = 1 + b_1 z + b_2 z^2 + \cdots$ and $b_n \geq 0$ for all $n \geq 2$.

Then have the following equivalence:

(a) $f \in \mathcal{S}$

(b) $\frac{f(z)f'(z)}{z} \neq 0$ for $z \in \Delta$

(c) $\sum_{n=2}^{\infty}(n-1)b_n \leq 1$

(d) $f \in \mathcal{U}$.

Theorem 6(Ali, Ob.,Po., 2011)

Let $f \in \mathcal{A}$ be such that $f(z) \neq 0$ for $0 < |z| < 1$ and have the form $\frac{z}{f(z)} = 1 + b_1 z + b_2 z^2 + \cdots$ satisfying the condition $\sum_{n=2}^{\infty}(n-1)|b_n|^2 \leq 1$. Then f is univalent in the disk $|z| < r_0 = \frac{1}{\sqrt{2}}$ and the result is the best possible.

Theorem 7(Ali, Ob.,Po., 2011)

Let $f \in \mathcal{A}$ be such that $f(z) \neq 0$ for $0 < |z| < 1$ and have the form $\frac{z}{f(z)} = 1 + b_1 z + b_2 z^2 + \cdots$ satisfying the condition $\sum_{n=2}^{\infty}(n-1)^2|b_n|^2 \leq 1$. Then the function g, defined by $g(z) = r^{-1}f(rz)$, belongs to $\mathcal{U}$ for $0 < r \leq r_0 = \sqrt{\frac{\sqrt{5}-1}{2}} \approx 0.78615$. In particular, f is univalent in the disk $|z| < r_0$ and the result is the best possible.

Theorem 8(Ob.,Po., Tuneski,2011)

Let $f, g \in \mathcal{S}$. Then the function F defined by $F(z) = \frac{zf(z)}{g(z)}$ is univalent in the disk $|z| < r_0$, where $r_0 = 0.21734\cdots$ is the root of the equation $20r^5 + 16r^4 - 23r^3 - 7r^2 + 7r - 1 = 0$ in the interval $(0, 1)$.

Conjecture

Suppose that $f, g \in \mathcal{S}$. Then the function F defined by $F(z) = \frac{zf(z)}{g(z)}$ is univalent in the disk $|z| < \sqrt{5} - 2 \approx 0.236068$.

Example

Let consider $f(z) = \frac{z}{(1+z)^2}, g(z) = \frac{z}{(1-z)^2}$ Then, we have $\frac{z}{F(z)} = \frac{g(z)}{f(z)} = \left(\frac{1+z}{1-z}\right)^2 = (1+2z+z^2)(1+2z+3z^2+\cdots)$ $= 1+4z+4\sum_{n=2}^{\infty} nz^n$. Since the Taylor coefficients of $z/F(z)$ are all positive and $\frac{rz}{F(rz)} = 1+4rz+4\sum_{n=2}^{\infty} nr^n z^n$, then according to Th.5, $\frac{1}{r}F(rz)$ is univalent in $\mathbb{D}$ if and only if $\sum_{n=2}^{\infty}(n-1)nr^n = \frac{8r^2}{(1-r)^3} \leq 1$. The last inequality is equivalent to $r^3+5r^2+3r-1 = (r+1)(r^2+4r-1) \leq 0$ which gives $0 < r \leq r_0 = \sqrt{5}-2 \approx 0.236068$, where r_0 is the unique positive root of the equation $r^2+4r-1 = 0$ in the interval $(0,1)$. Thus, F is univalent in the disk $|z| < \sqrt{5}-2$.

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THANK YOU