From Applications to the Theory: A Combinatorial Approach to Matrix Theory

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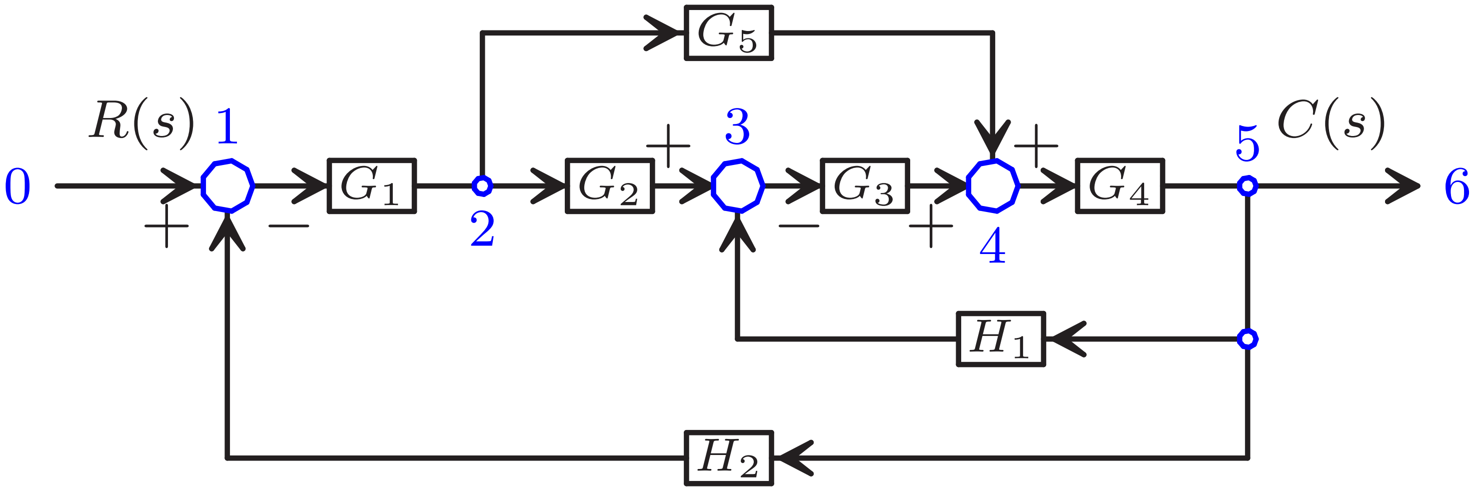
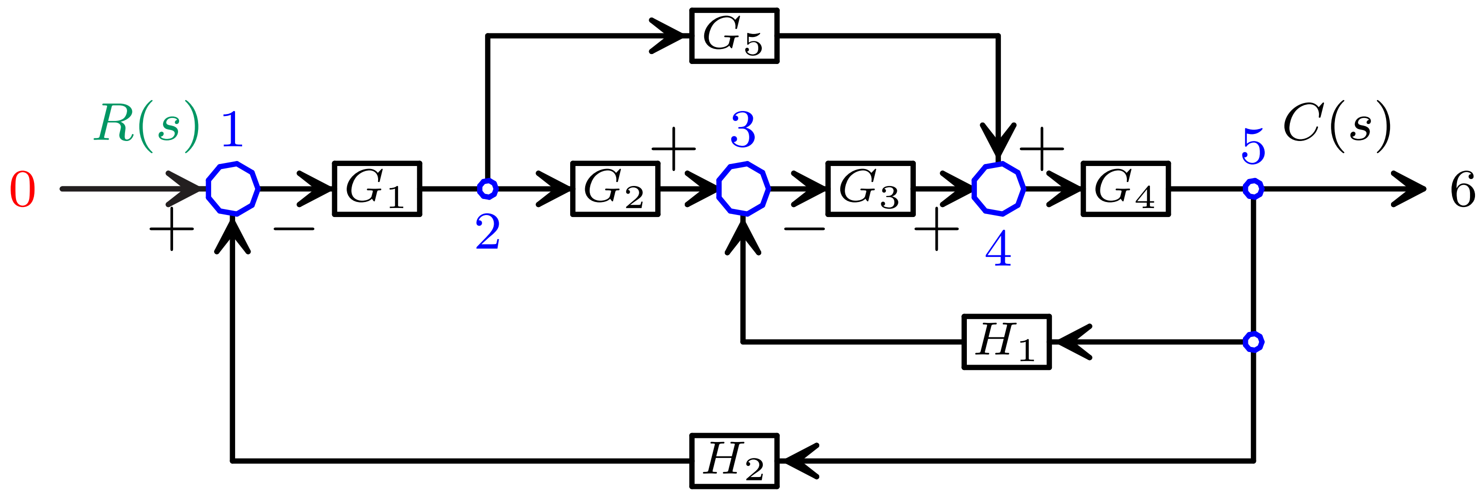
During the Second Word War C.E.Shannon invented a graph theoretical procedure for solving systems of linear algebraic equations as an aid in designing weapon control systems. This technique was for some time a military secret but became later known as signal flow graph technique, nowadays widely used by electrical engineers in control theory.



Block diagram

$$
\begin{aligned}
x_0 &= R(s) & x_1 &= x_0 - H_2 x_5 \\
x_2 &= G_1 x_1 & x_3 &= G_2 x_2 - H_1 x_5 \\
x_4 &= G_3 x_3 + G_5 x_2 & x_5 &= G_4 x_4 \\
x_6 &= C(s)
\end{aligned}
$$



Equations

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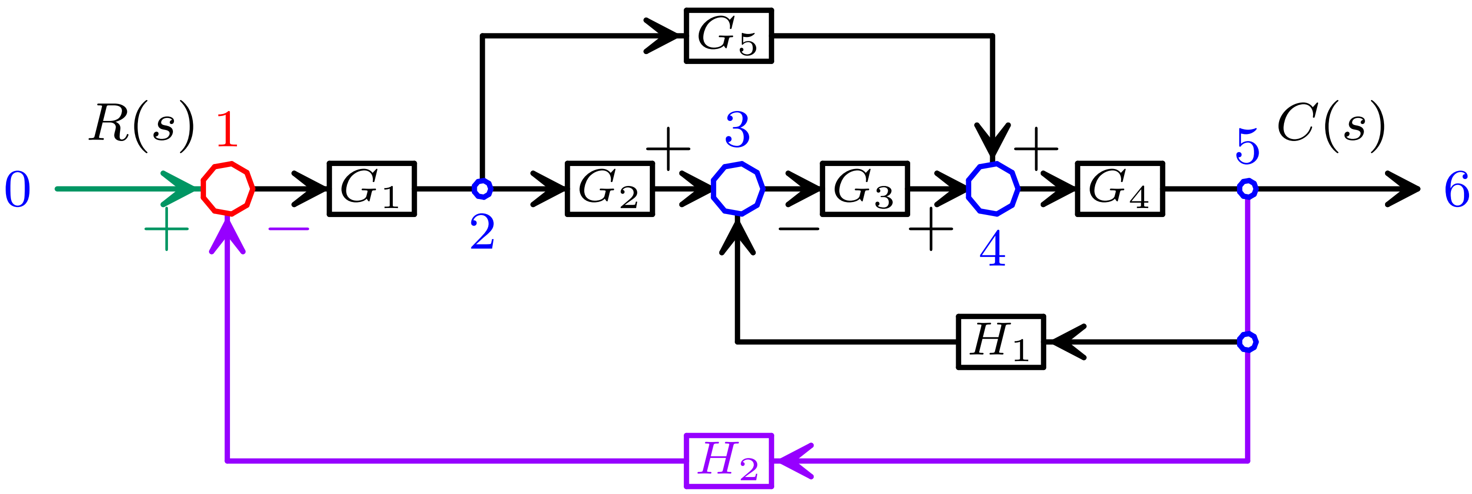

Equations

$$x_0 = R(s) \qquad x_1 = x_0 - H_2 x_5$$
$$x_2 = G_1 x_1 \qquad x_3 = G_2 x_2 - H_1 x_5$$
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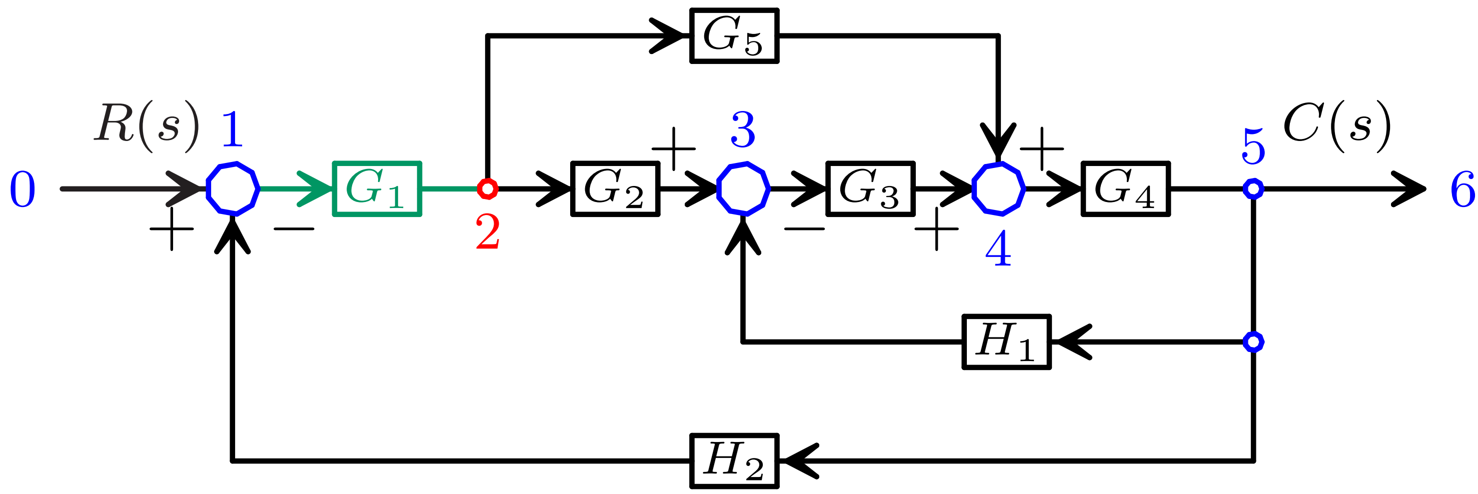
$$x_6 = C(s)$$



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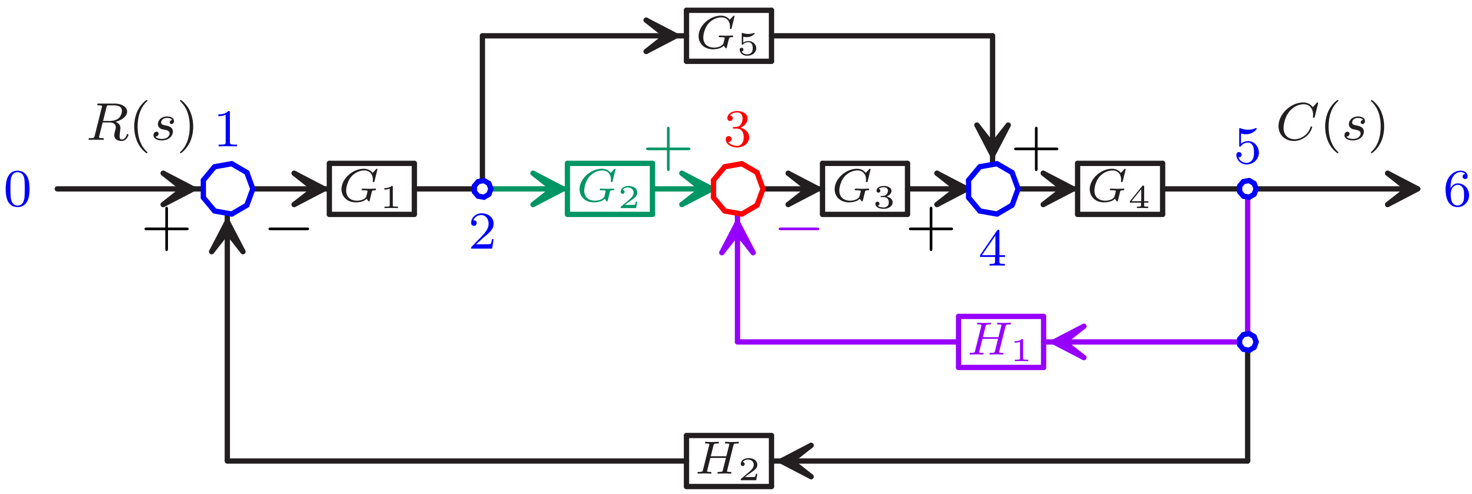

Equations

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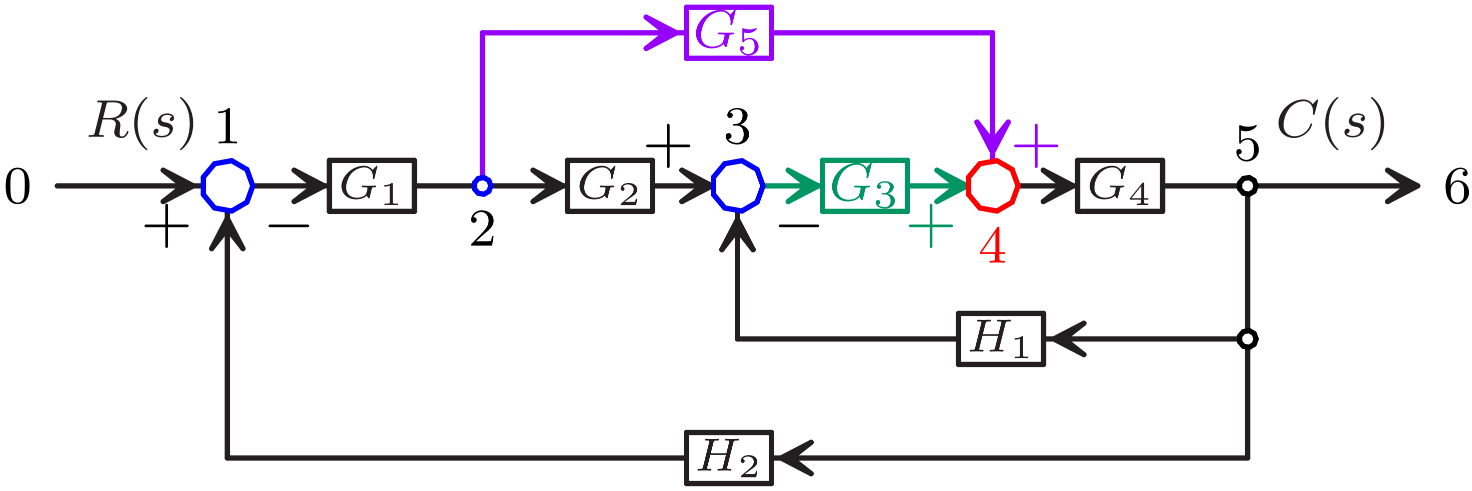

Equations

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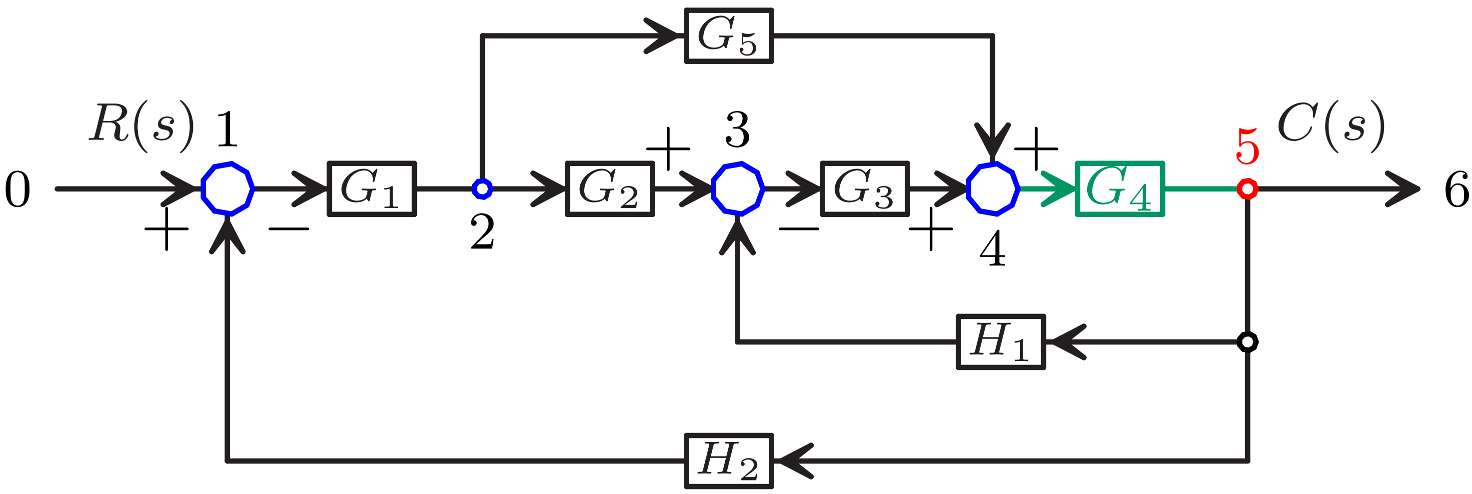

Equations

$$x_0 = R(s) \qquad x_1 = x_0 - H_2 x_5$$

$$x_2 = G_1 x_1 \qquad x_3 = G_2 x_2 - H_1 x_5$$

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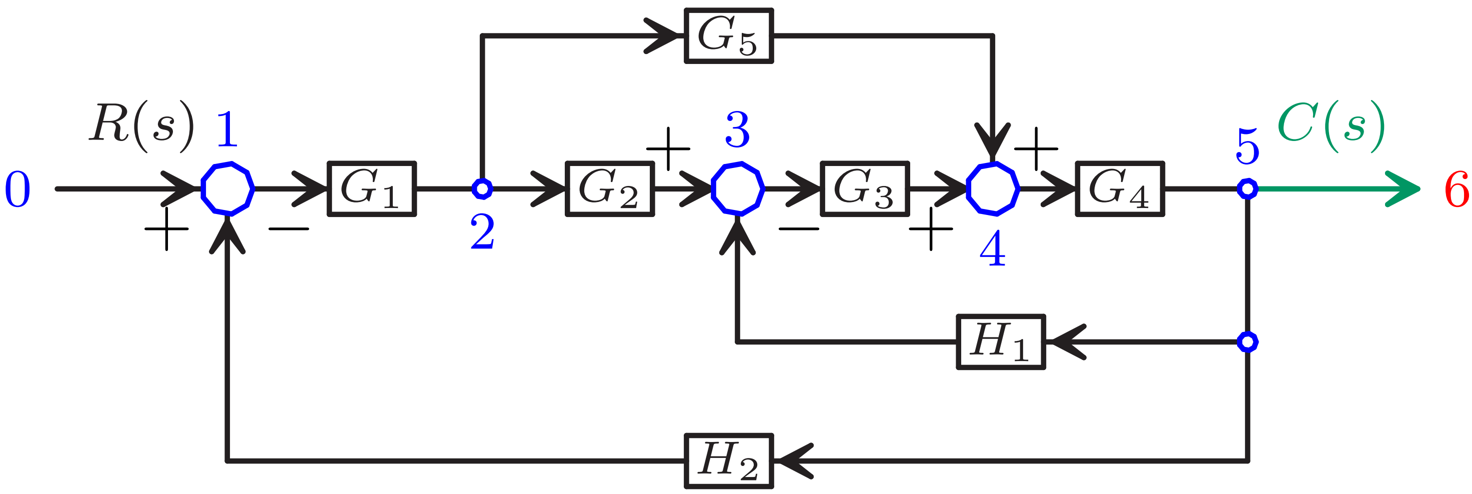

Equations

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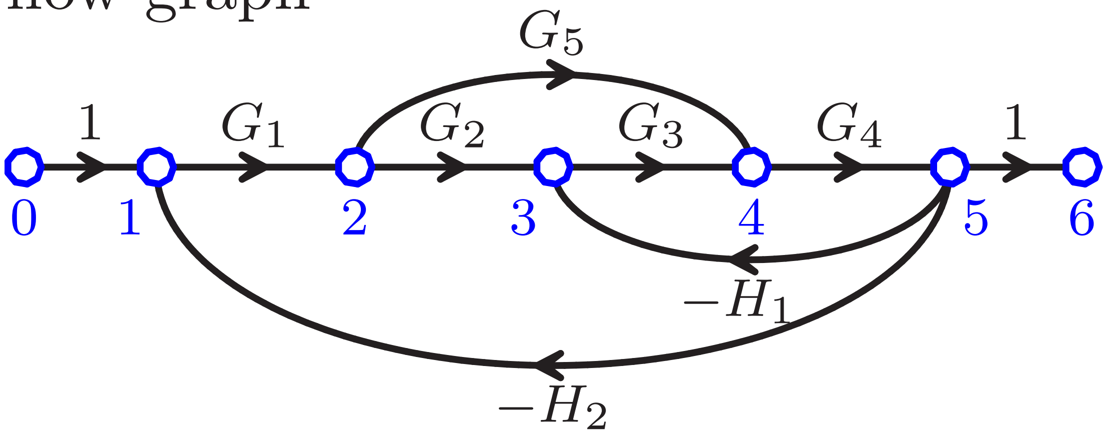
x_6 &= C(s) & &
\end{aligned}
$$

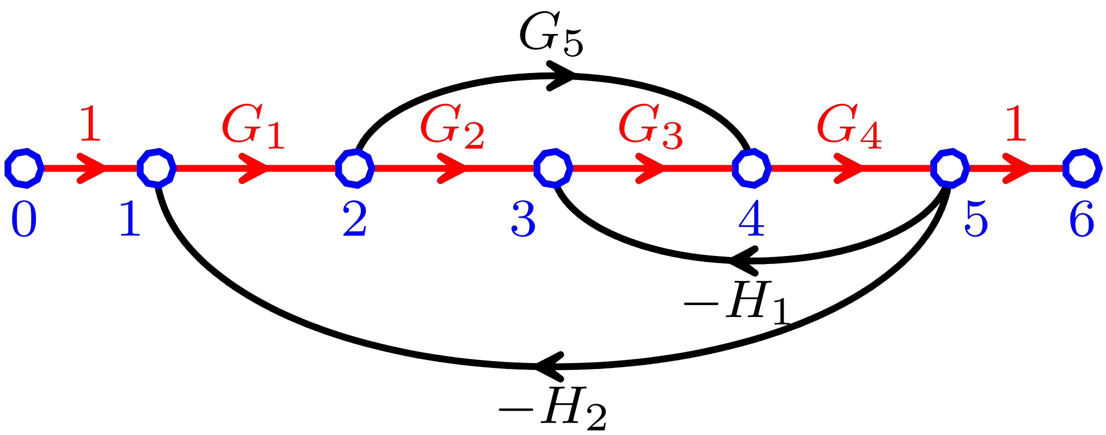
Signal flow graph



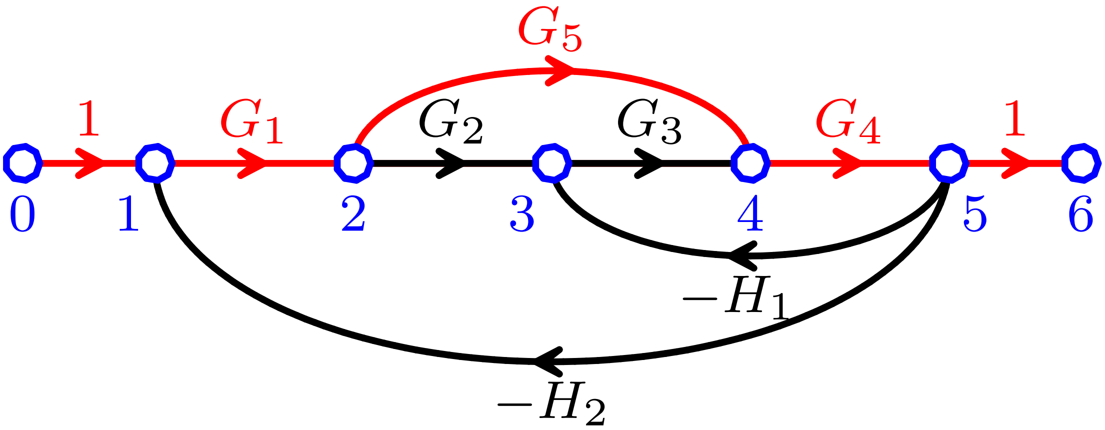
Mason's formula

S.J.Mason, 1953

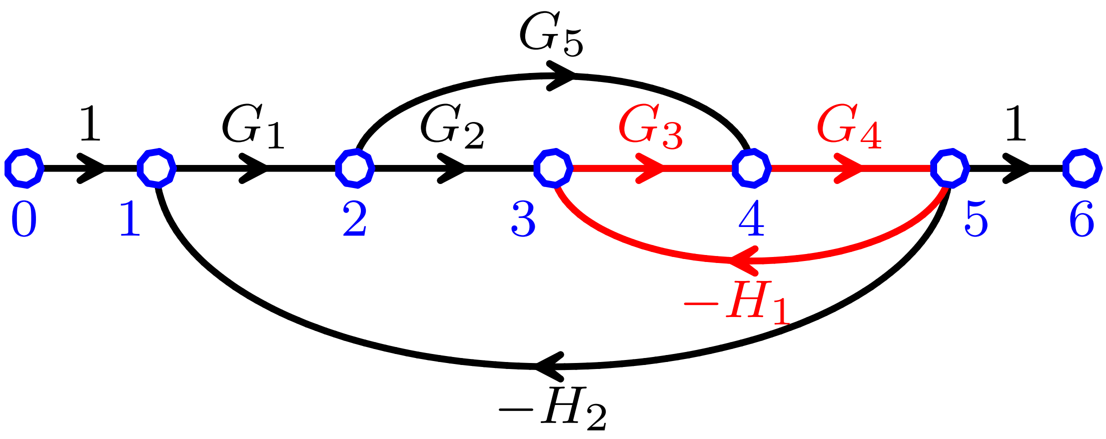
$$\frac{C(s)}{R(s)} = \frac{G_1G_2G_3G_4 + G_1G_5G_4}{1 + G_3G_4H_1 + G_1G_2G_3G_4H_2 + G_1G_5G_4H_2}$$



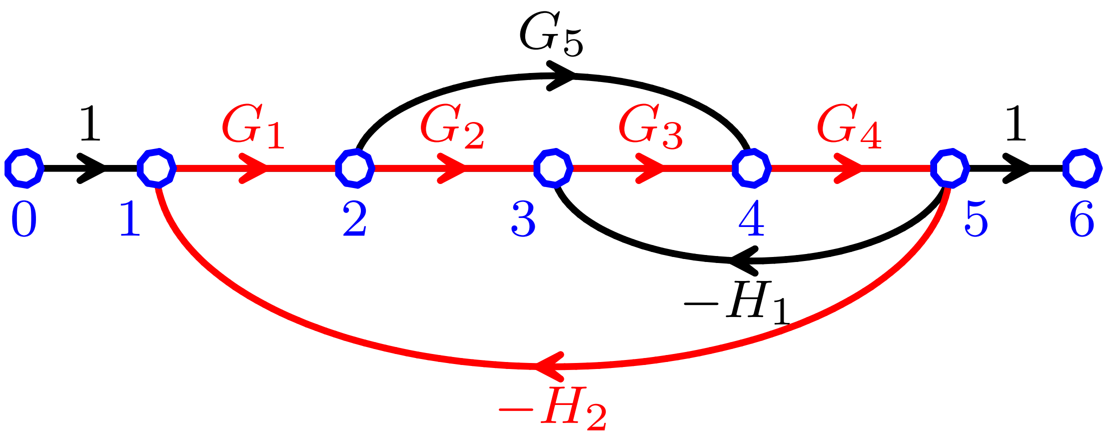
$$\frac{C(s)}{R(s)} = \frac{1G_1G_2G_3G_41 + 1G_1G_5G_41}{1 + G_3G_4H_1 + G_1G_2G_3G_4H_2 + G_1G_5G_4H_2}$$



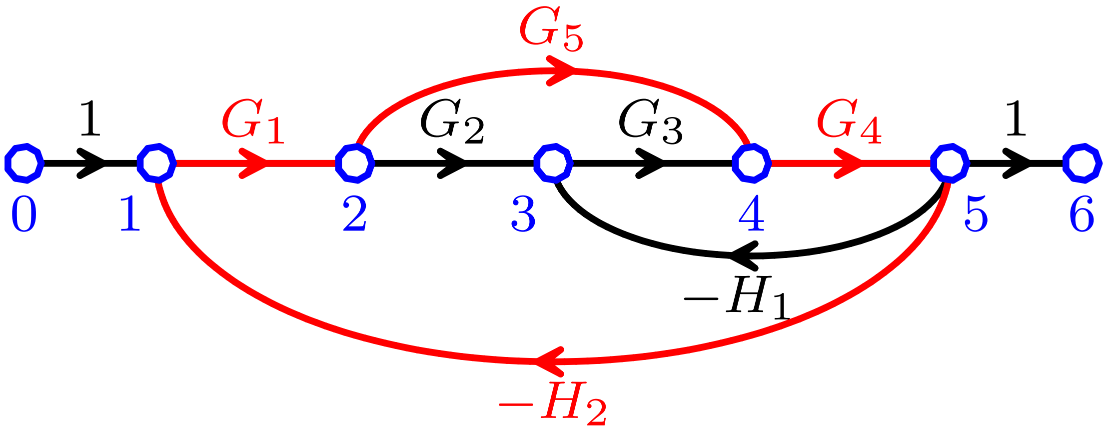
$$\frac{C(s)}{R(s)} = \frac{1G_1G_2G_3G_41 + 1G_1G_5G_41}{1 + G_3G_4H_1 + G_1G_2G_3G_4H_2 + G_1G_5G_4H_2}$$



$$\frac{C(s)}{R(s)} = \frac{1G_1G_2G_3G_41 + 1G_1G_5G_41}{1 + G_3G_4H_1 + G_1G_2G_3G_4H_2 + G_1G_5G_4H_2}$$



$$\frac{C(s)}{R(s)} = \frac{1G_1G_2G_3G_41 + 1G_1G_5G_41}{1 + G_3G_4H_1 + G_1G_2G_3G_4H_2 + G_1G_5G_4H_2}$$



$$\frac{C(s)}{R(s)} = \frac{1G_1G_2G_3G_4 1 + 1G_1G_5G_4 1}{1 + G_3G_4H_1 + G_1G_2G_3G_4H_2 + G_1G_5G_4H_2}$$

These facts, together with several results in combinatorial matrix theory, theory of graph spectra, theory of sparse matrices and theoretical chemistry have led to a combinatorial approach to classical elementary matrix theory. This approach is described in the book just published:

Brualdi R.A., Cvetković D.,

A Combinatorial Approach to Matrix Theory and Its Application,

CRC Press, Boca Raton, 2008.

We present basic ideas of this book.

Previous books:

- R. A. Brualdi, H. J. Ryser, *Combinatorial Matrix Theory*, Cambridge University Press, Cambridge, 1991; reprinted 1992.
- D. Cvetković, *Combinatorial Matrix Theory, with Applications to Electrical Engineering, Chemistry and Physics*, (in Serbian), Naučna knjiga, Beograd, 1980; II edition 1987.

empty scheme

$$\begin{bmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{bmatrix}$$

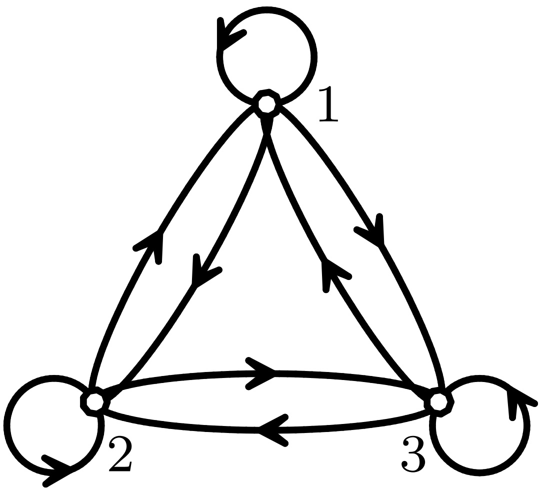
empty scheme $\qquad\qquad$ matrix

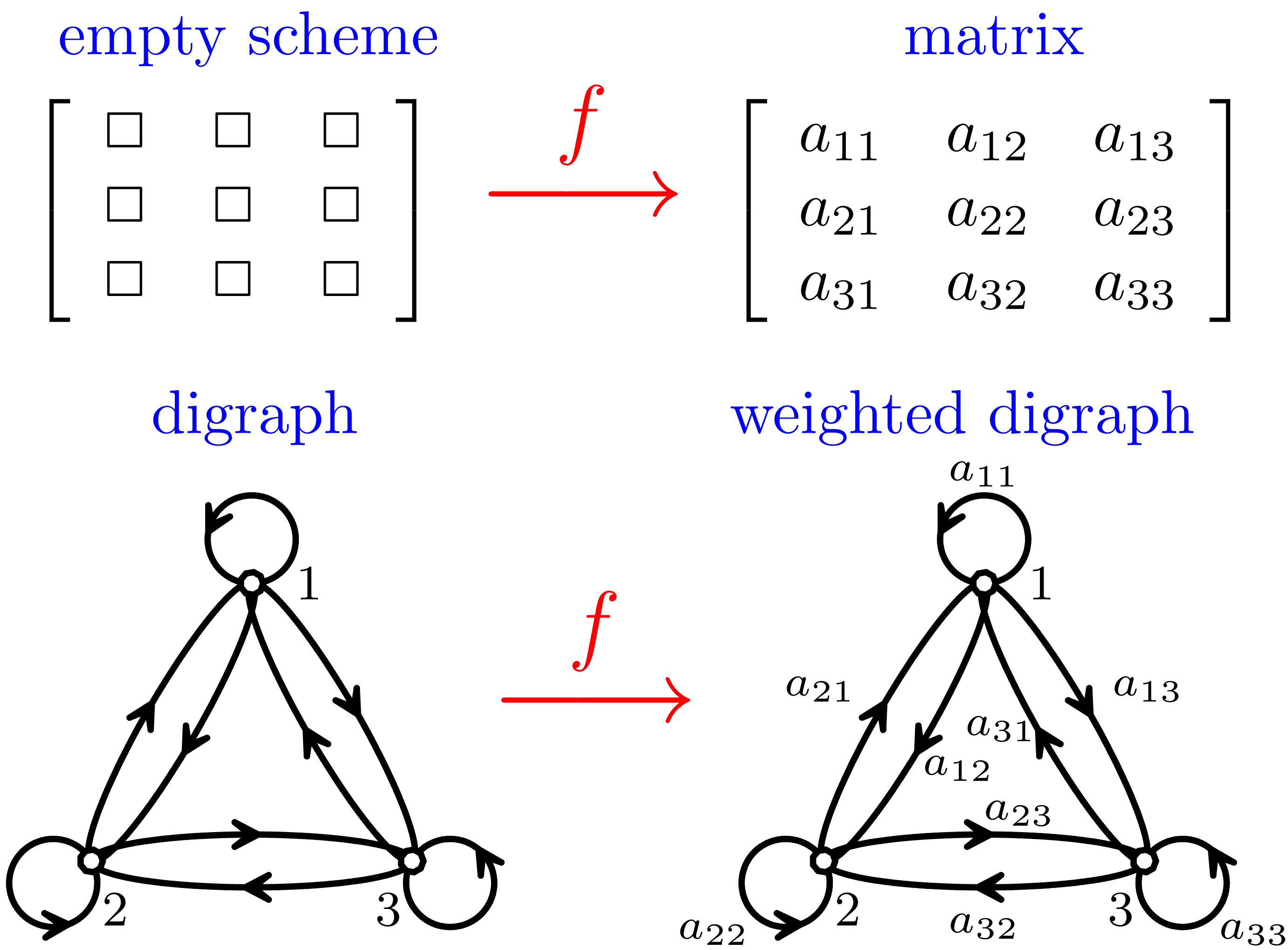
$$\begin{bmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{bmatrix} \xrightarrow{\quad f \quad} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

empty scheme   matrix

$$\begin{bmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{bmatrix} \xrightarrow{f} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

digraph





empty scheme
matrix
f
a_{11} a_{12} a_{13}
a_{21} a_{22} a_{23}
a_{31} a_{32} a_{33}
digraph
weighted digraph
1
2
3
f
a_{11}
a_{21}
a_{13}
a_{31}
a_{12}
a_{23}
a_{22}
a_{32}
a_{33}

Let $A = [a_{ij}]$ be a square matrix of order n. The *determinant* of A is the number $\det A$ defined by the sum

$$\det A = (-1)^n \sum_{L \in \mathcal{L}(A)} (-1)^{c(L)} w(L)$$

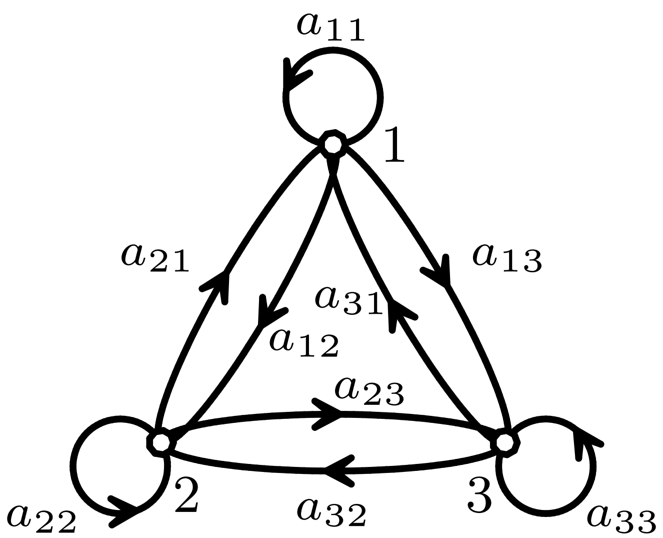
where the summation extends over all linear subdigraphs L of the digraph $D^*(A)$.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

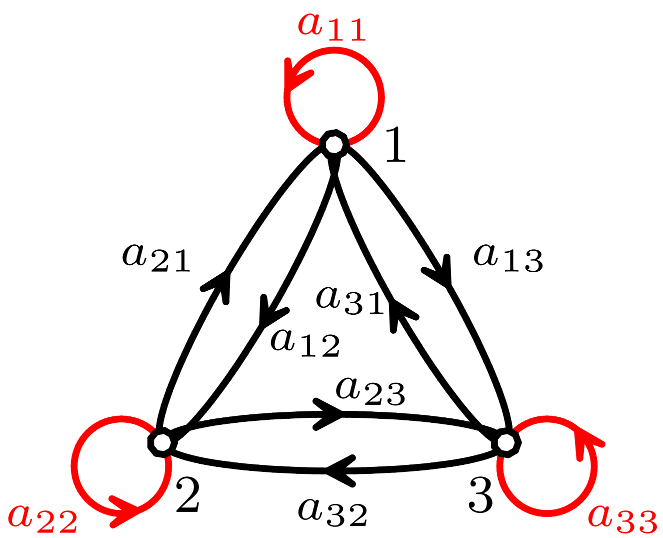
$$\begin{aligned} \det A &= (-1)^{3+3} a_{11}a_{22}a_{33} + (-1)^{3+1} a_{12}a_{31}a_{23} + \\ &\quad (-1)^{3+1} a_{21}a_{32}a_{13} + (-1)^{3+2} a_{11}a_{23}a_{32} + \\ &\quad (-1)^{3+2} a_{22}a_{13}a_{31} + (-1)^{3+2} a_{33}a_{12}a_{21} \end{aligned}$$

$$\begin{aligned} \det A &= a_{11}a_{22}a_{33} + a_{12}a_{31}a_{23} + a_{21}a_{32}a_{13} \\ &= -a_{11}a_{23}a_{32} - a_{22}a_{13}a_{31} - a_{33}a_{12}a_{21} \end{aligned}$$

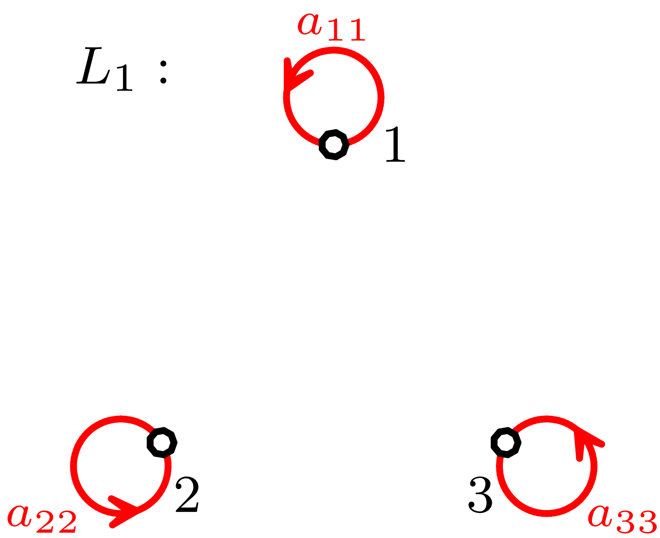
$$
\begin{aligned}
\det A \quad = \quad & (-1)^{3+3} a_{11} a_{22} a_{33} + (-1)^{3+1} a_{12} a_{31} a_{23} + \\
& (-1)^{3+1} a_{21} a_{32} a_{13} + (-1)^{3+2} a_{11} a_{23} a_{32} + \\
& (-1)^{3+2} a_{22} a_{13} a_{31} + (-1)^{3+2} a_{33} a_{12} a_{21}
\end{aligned}
$$



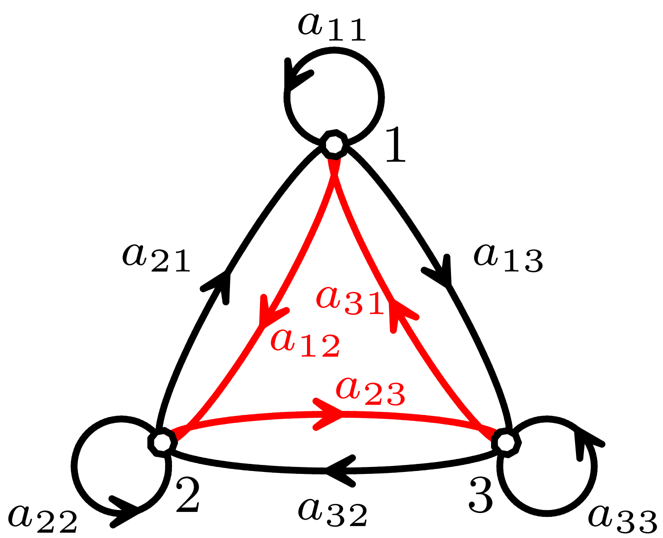
$$\begin{aligned}\det A \quad = \quad & (-1)^{3+3} a_{11} a_{22} a_{33} + (-1)^{3+1} a_{12} a_{31} a_{23} + \\ & (-1)^{3+1} a_{21} a_{32} a_{13} + (-1)^{3+2} a_{11} a_{23} a_{32} + \\ & (-1)^{3+2} a_{22} a_{13} a_{31} + (-1)^{3+2} a_{33} a_{12} a_{21}\end{aligned}$$



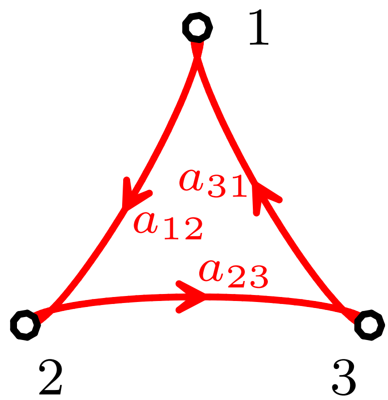
L_1 :



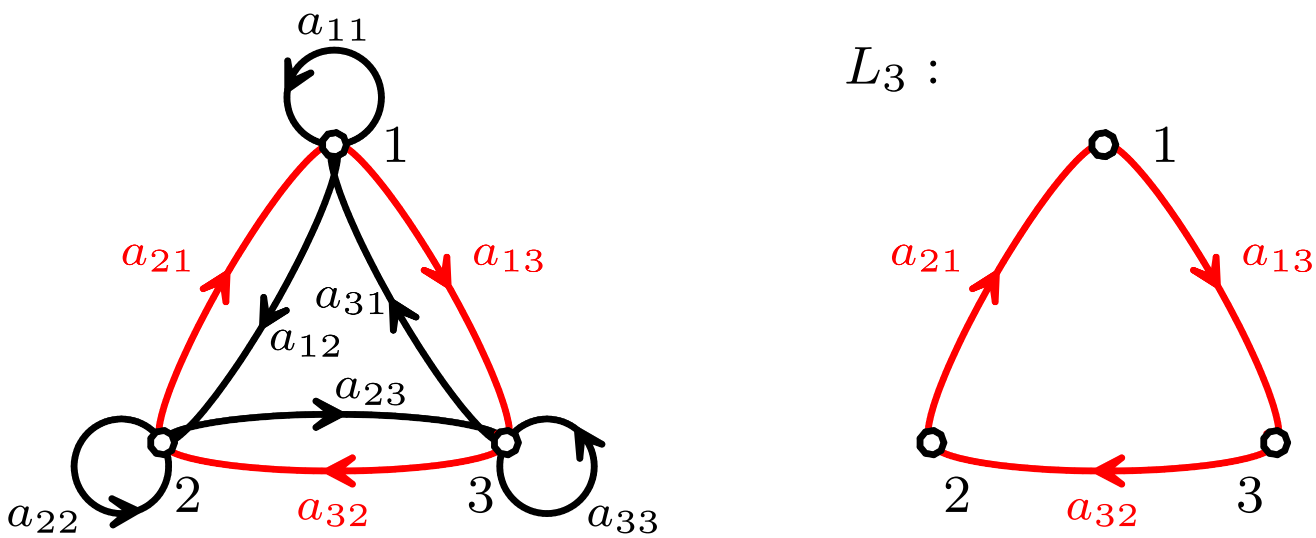
$$\begin{aligned}\det A \quad = \quad & (-1)^{3+3}a_{11}a_{22}a_{33} + (-1)^{3+1}a_{12}a_{31}a_{23} + \\ & (-1)^{3+1}a_{21}a_{32}a_{13} + (-1)^{3+2}a_{11}a_{23}a_{32} + \\ & (-1)^{3+2}a_{22}a_{13}a_{31} + (-1)^{3+2}a_{33}a_{12}a_{21}\end{aligned}$$



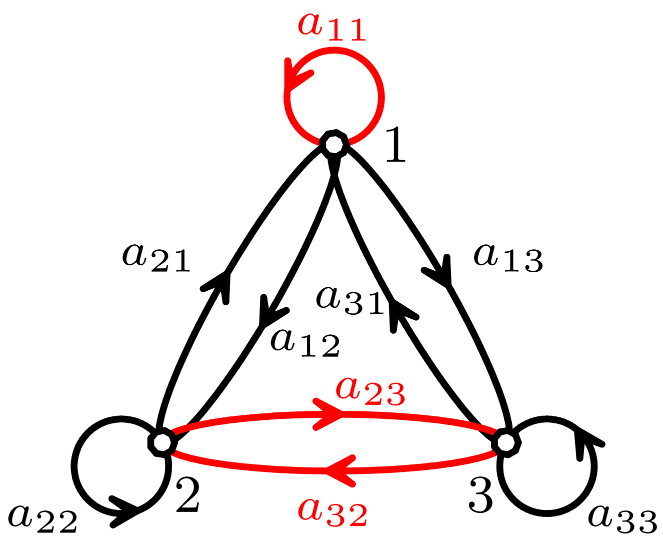
L_2 :

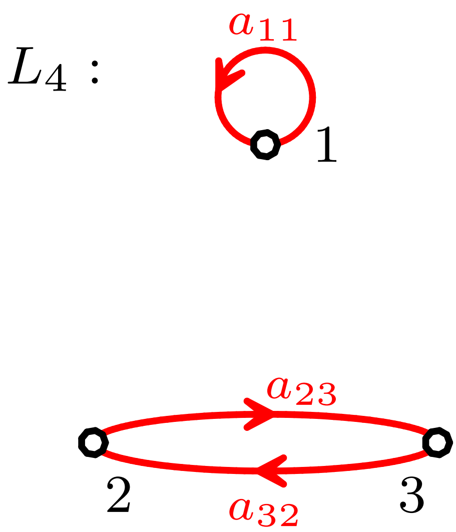


$$
\begin{aligned}
\det A \quad = \quad & (-1)^{3+3} a_{11}a_{22}a_{33} + (-1)^{3+1} a_{12}a_{31}a_{23} + \\
& (-1)^{3+1} a_{21}a_{32}a_{13} + (-1)^{3+2} a_{11}a_{23}a_{32} + \\
& (-1)^{3+2} a_{22}a_{13}a_{31} + (-1)^{3+2} a_{33}a_{12}a_{21}
\end{aligned}
$$

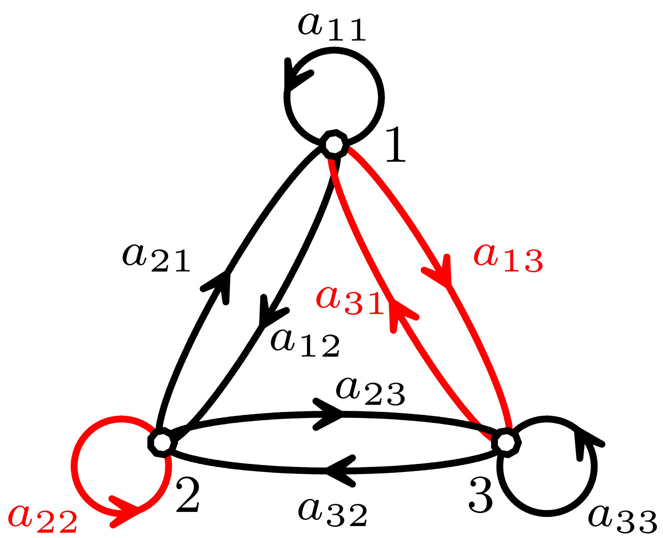


$$
\begin{aligned}
\det A \quad = \quad & (-1)^{3+3} a_{11} a_{22} a_{33} + (-1)^{3+1} a_{12} a_{31} a_{23} + \\
& (-1)^{3+1} a_{21} a_{32} a_{13} + (-1)^{3+2} a_{11} a_{23} a_{32} + \\
& (-1)^{3+2} a_{22} a_{13} a_{31} + (-1)^{3+2} a_{33} a_{12} a_{21}
\end{aligned}
$$

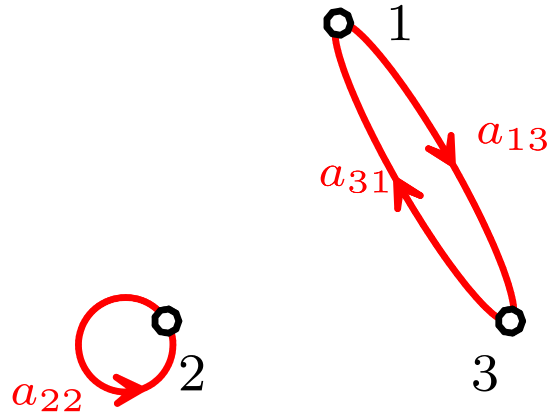




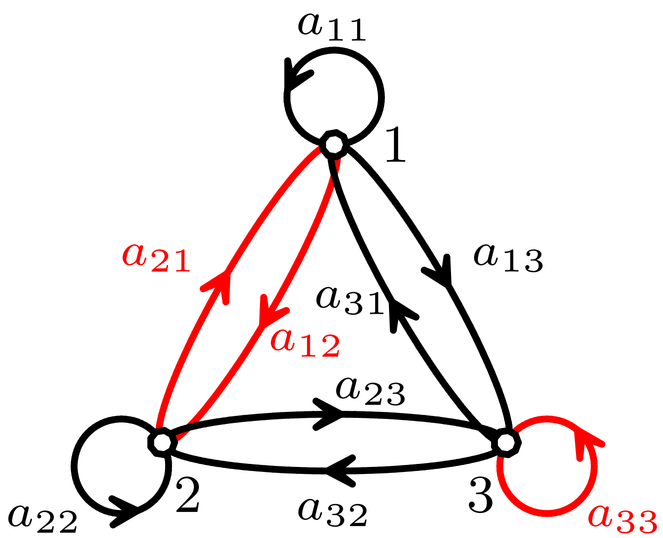
$$
\begin{aligned}
\det A \quad &= \quad (-1)^{3+3}a_{11}a_{22}a_{33} + (-1)^{3+1}a_{12}a_{31}a_{23} + \\
&\quad\ (-1)^{3+1}a_{21}a_{32}a_{13} + (-1)^{3+2}a_{11}a_{23}a_{32} + \\
&\quad\ (-1)^{3+2}a_{22}a_{13}a_{31} + (-1)^{3+2}a_{33}a_{12}a_{21}
\end{aligned}
$$



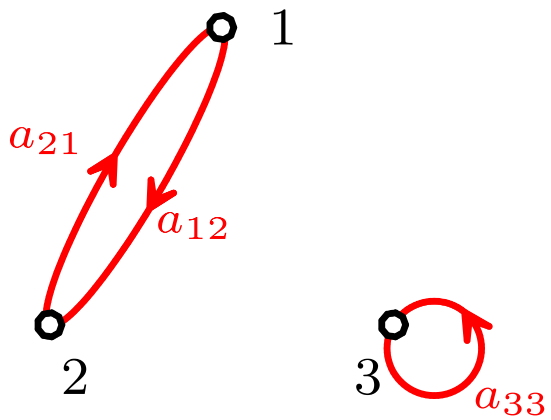
L_5 :



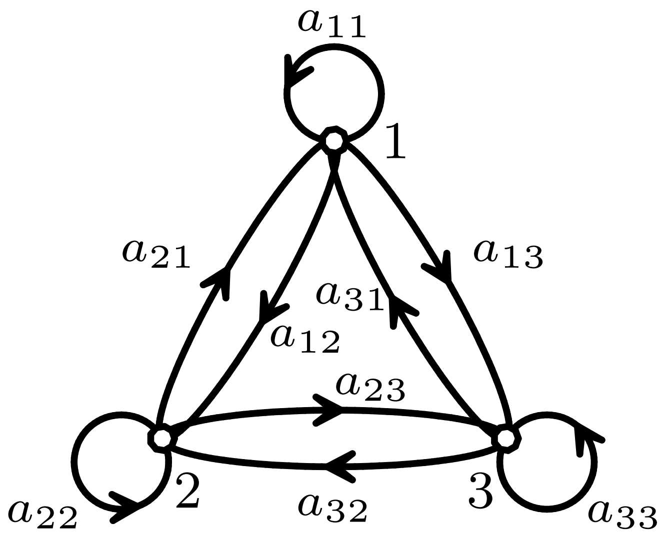
$$
\begin{aligned}
\det A \quad = \quad & (-1)^{3+3} a_{11} a_{22} a_{33} + (-1)^{3+1} a_{12} a_{31} a_{23} + \\
& (-1)^{3+1} a_{21} a_{32} a_{13} + (-1)^{3+2} a_{11} a_{23} a_{32} + \\
& (-1)^{3+2} a_{22} a_{13} a_{31} + (-1)^{3+2} a_{33} a_{12} a_{21}
\end{aligned}
$$

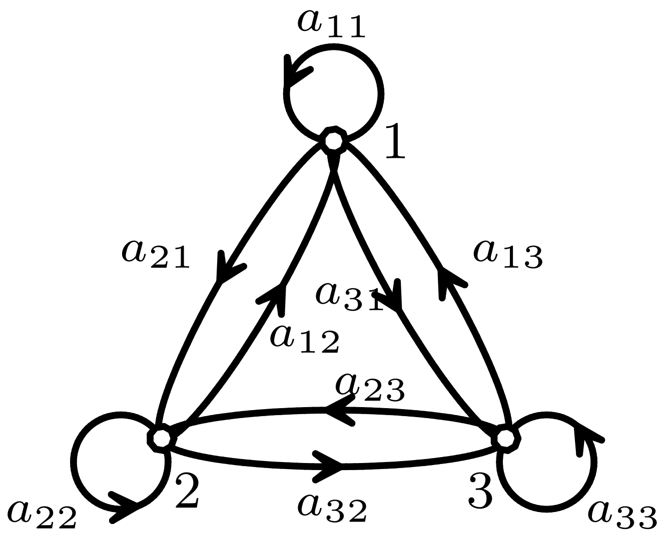


L_6 :



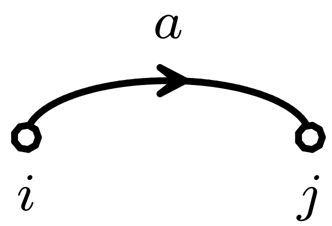
$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad A^\top = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$





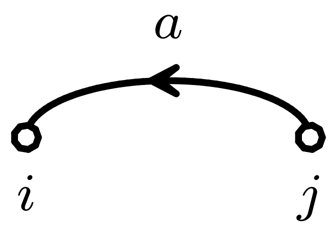
Effect of transposition

entry a at position (i, j)



Effect of transposition

entry a at position (j, i)

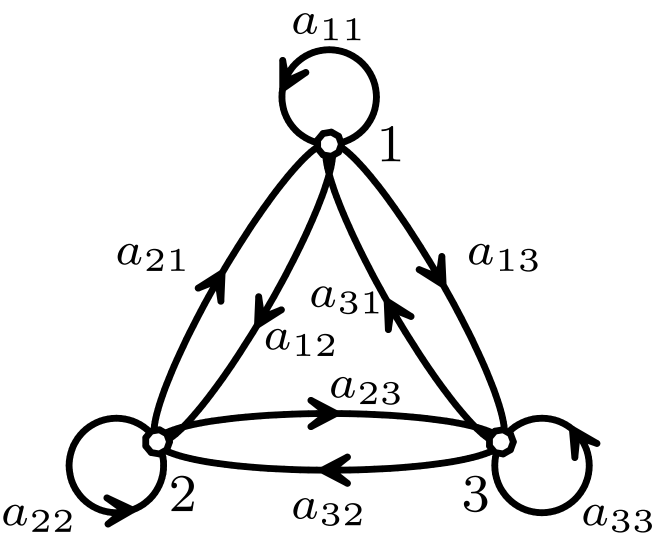


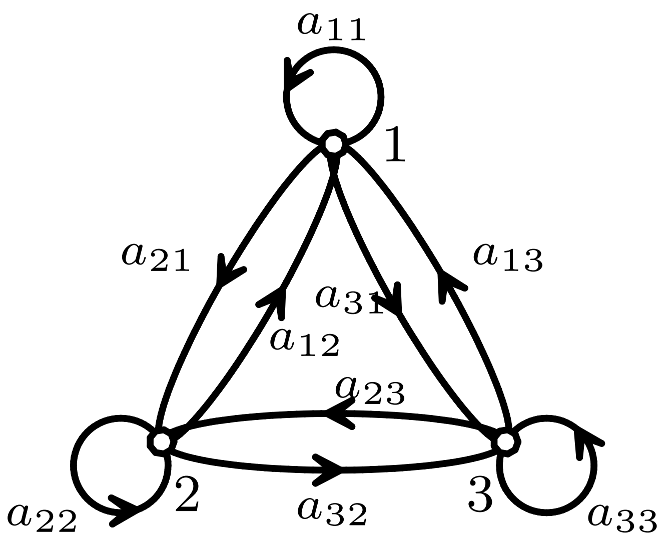
$$\det A = \qquad\qquad \det A^\top =$$

$$= (-1)^{3+3} a_{11}a_{22}a_{33} + (-1)^{3+1} a_{12}a_{31}a_{23}$$

$$+(-1)^{3+1} a_{21}a_{32}a_{13} + (-1)^{3+2} a_{11}a_{23}a_{32}$$

$$+(-1)^{3+2} a_{22}a_{13}a_{31} + (-1)^{3+2} a_{33}a_{12}a_{21}$$



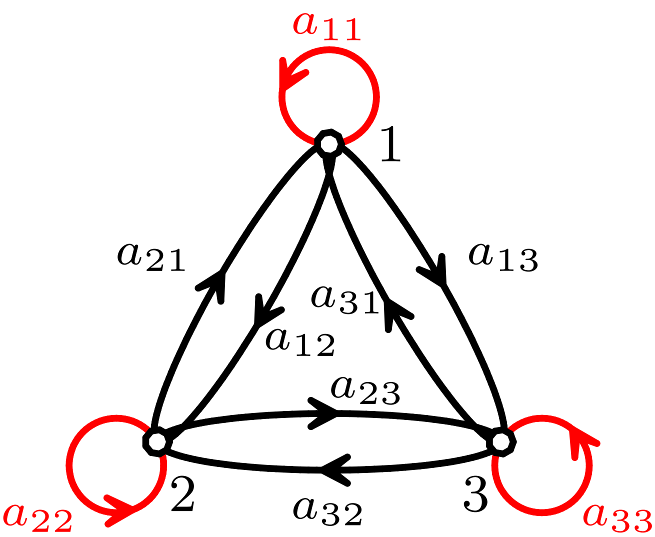


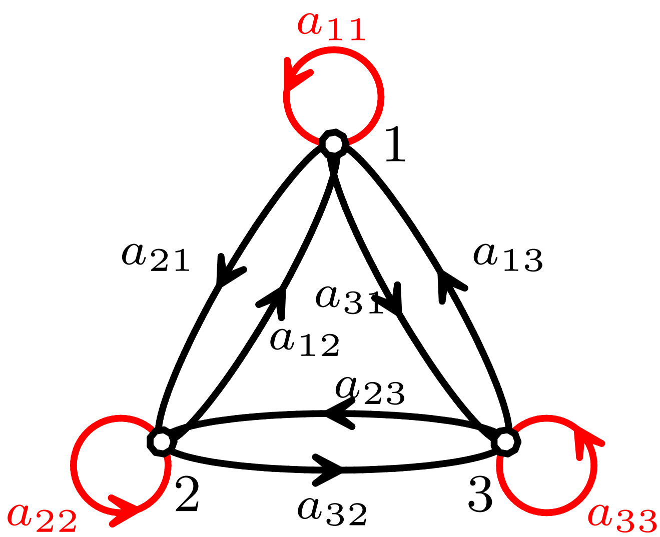
$$\det A = \qquad\qquad \det A^\top =$$

$$= {\color{red}(-1)^{3+3} a_{11}a_{22}a_{33}} + (-1)^{3+1} a_{12}a_{31}a_{23}$$

$$+(-1)^{3+1} a_{21}a_{32}a_{13} + (-1)^{3+2} a_{11}a_{23}a_{32}$$

$$+(-1)^{3+2} a_{22}a_{13}a_{31} + (-1)^{3+2} a_{33}a_{12}a_{21}$$



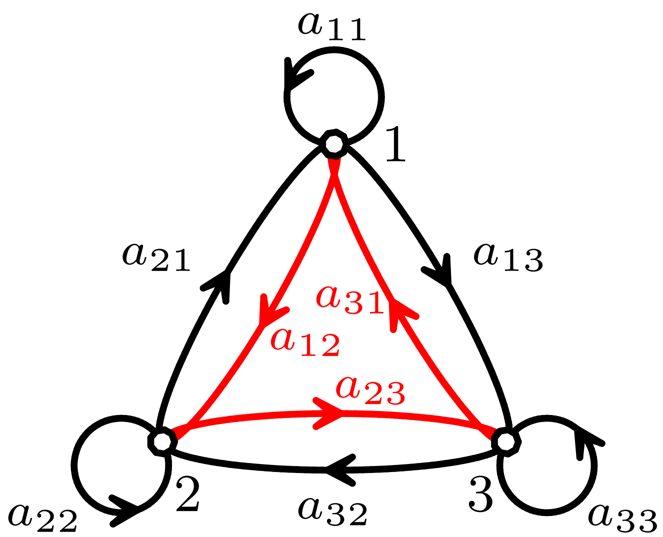


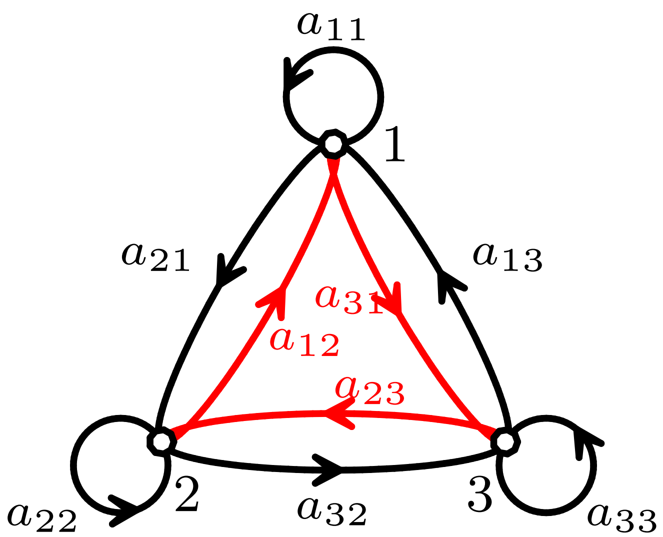
$\det A =$ $\qquad\qquad$ $\det A^\top =$

$$= (-1)^{3+3} a_{11}a_{22}a_{33} + (-1)^{3+1} a_{12}a_{31}a_{23}$$

$$+(-1)^{3+1} a_{21}a_{32}a_{13} + (-1)^{3+2} a_{11}a_{23}a_{32}$$

$$+(-1)^{3+2} a_{22}a_{13}a_{31} + (-1)^{3+2} a_{33}a_{12}a_{21}$$



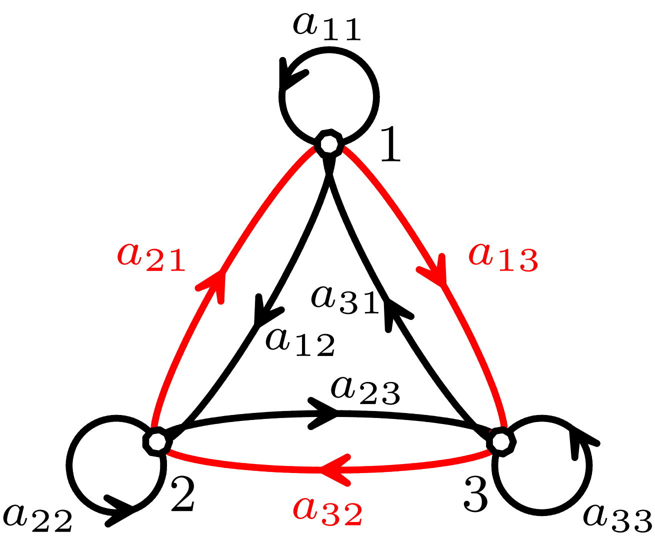


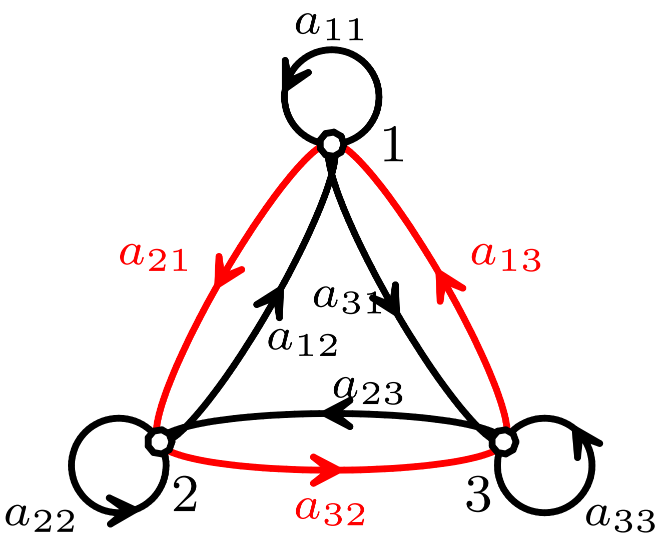
$\det A =$ $\qquad\qquad$ $\det A^{\top} =$

$$= (-1)^{3+3} a_{11}a_{22}a_{33} + (-1)^{3+1} a_{12}a_{31}a_{23}$$

$$+(-1)^{3+1} a_{21}a_{32}a_{13} + (-1)^{3+2} a_{11}a_{23}a_{32}$$

$$+(-1)^{3+2} a_{22}a_{13}a_{31} + (-1)^{3+2} a_{33}a_{12}a_{21}$$



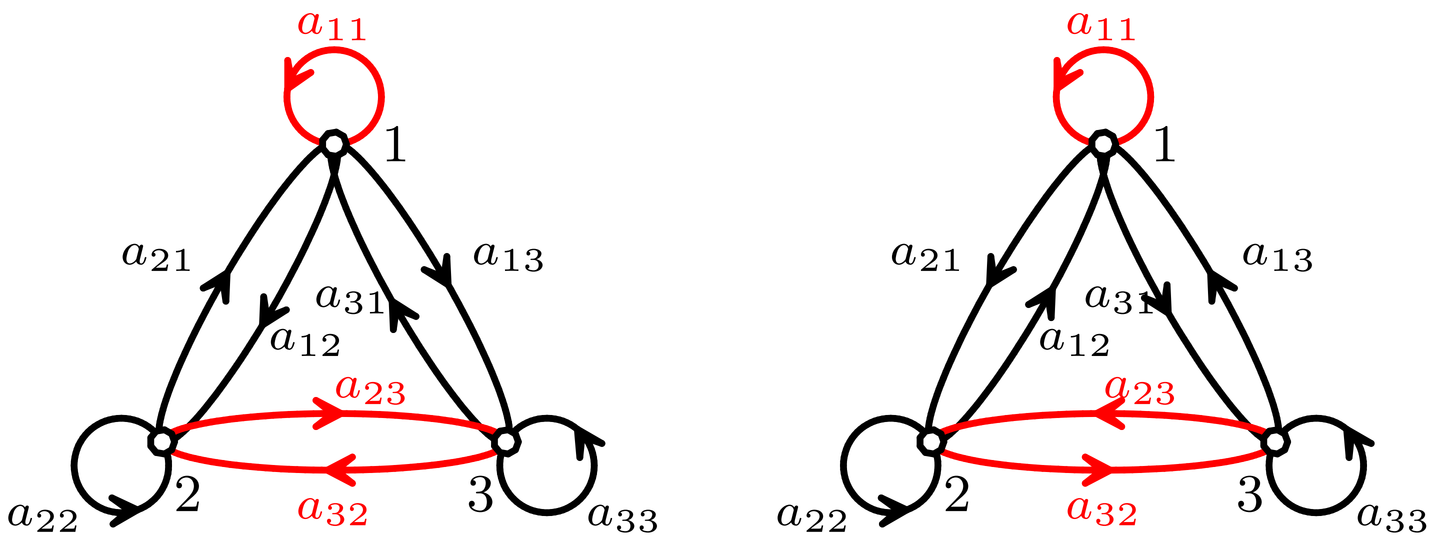


$$\det A = \qquad\qquad \det A^\top =$$

$$= (-1)^{3+3}a_{11}a_{22}a_{33} + (-1)^{3+1}a_{12}a_{31}a_{23}$$

$$+(-1)^{3+1}a_{21}a_{32}a_{13} + (-1)^{3+2}a_{11}a_{23}a_{32}$$

$$+(-1)^{3+2}a_{22}a_{13}a_{31} + (-1)^{3+2}a_{33}a_{12}a_{21}$$

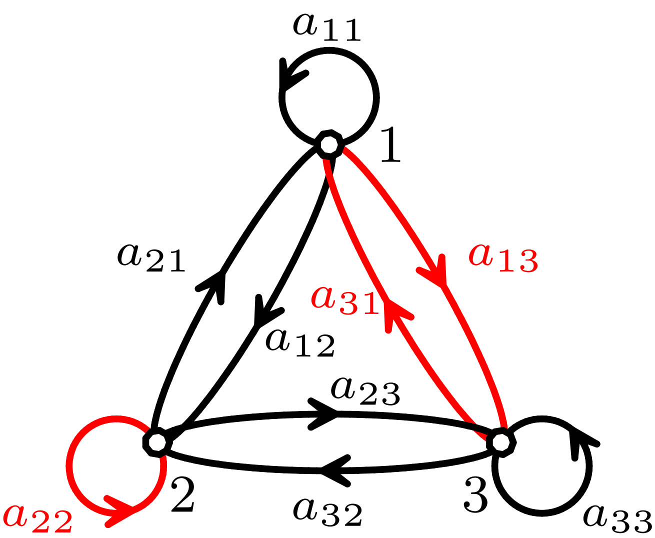


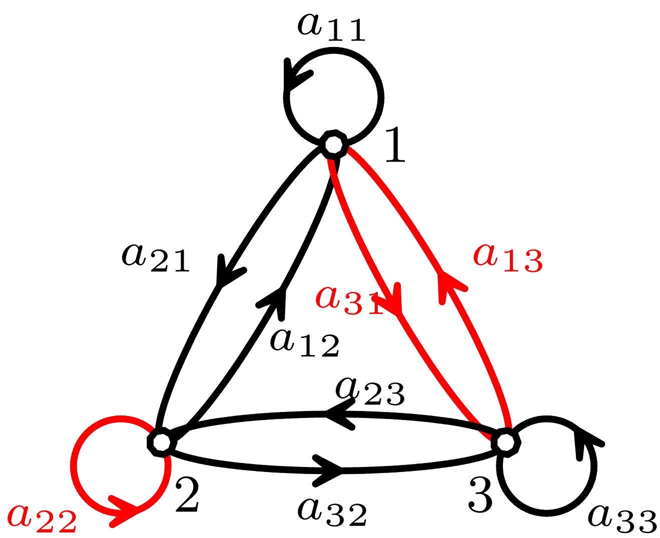
$$\det A = \qquad\qquad \det A^{\top} =$$

$$= (-1)^{3+3} a_{11} a_{22} a_{33} + (-1)^{3+1} a_{12} a_{31} a_{23}$$

$$+(-1)^{3+1} a_{21} a_{32} a_{13} + (-1)^{3+2} a_{11} a_{23} a_{32}$$

$$+(-1)^{3+2} a_{22} a_{13} a_{31} + (-1)^{3+2} a_{33} a_{12} a_{21}$$



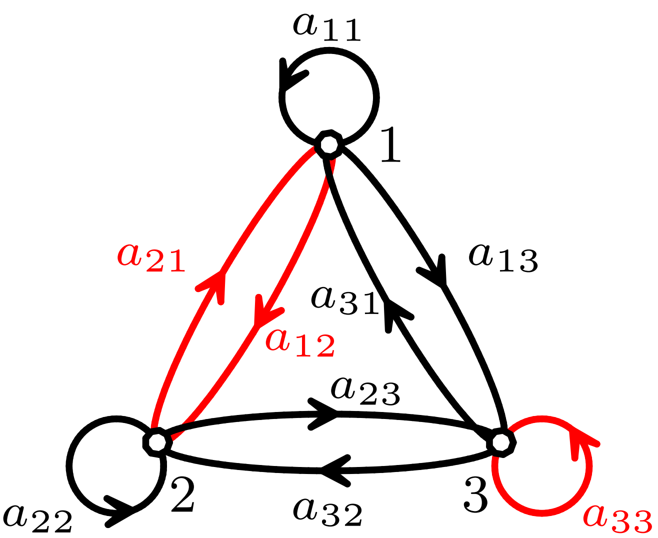


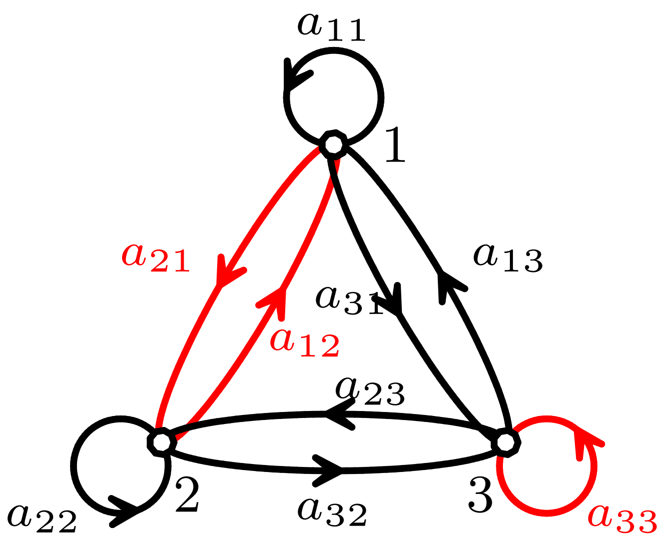
$\det A =$ $\qquad\qquad$ $\det A^\top =$

$$= (-1)^{3+3} a_{11}a_{22}a_{33} + (-1)^{3+1} a_{12}a_{31}a_{23}$$

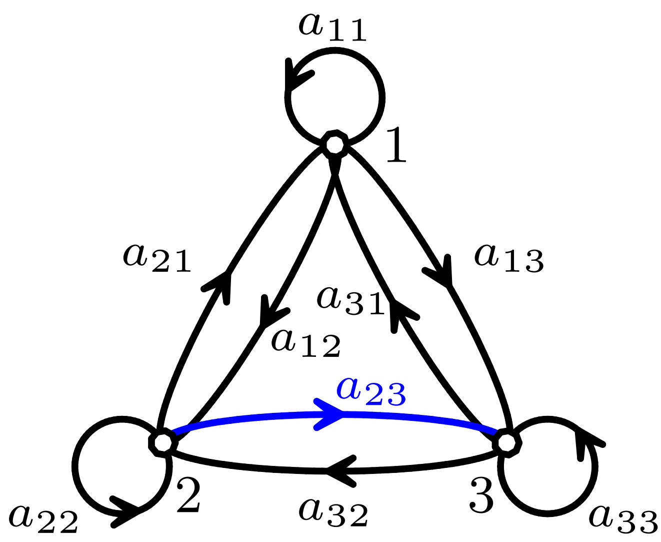
$$+(-1)^{3+1} a_{21}a_{32}a_{13} + (-1)^{3+2} a_{11}a_{23}a_{32}$$

$$+(-1)^{3+2} a_{22}a_{13}a_{31} + (-1)^{3+2} a_{33}a_{12}a_{21}$$



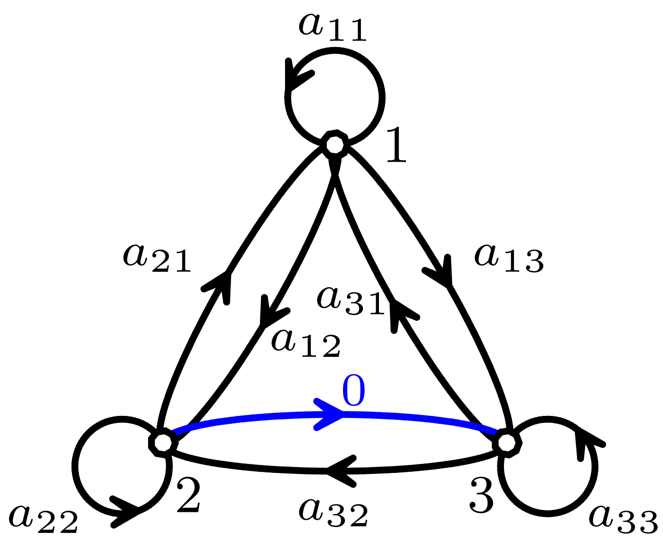


$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & \color{blue}{a_{23}} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$



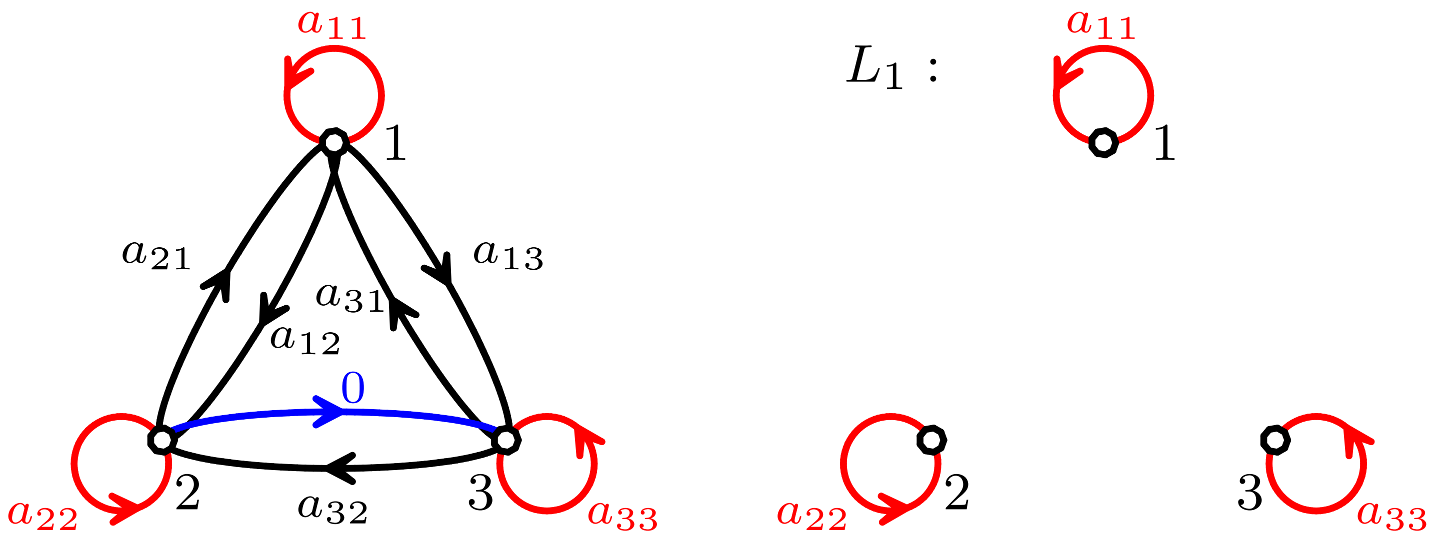
$$\begin{aligned} \det A \quad = \quad & (-1)^{3+3} a_{11} a_{22} a_{33} + (-1)^{3+1} a_{12} a_{31} \color{blue}{a_{23}} + \\ & (-1)^{3+1} a_{21} a_{32} a_{13} + (-1)^{3+2} a_{11} \color{blue}{a_{23}} a_{32} + \\ & (-1)^{3+2} a_{22} a_{13} a_{31} + (-1)^{3+2} a_{33} a_{12} a_{21} \end{aligned}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

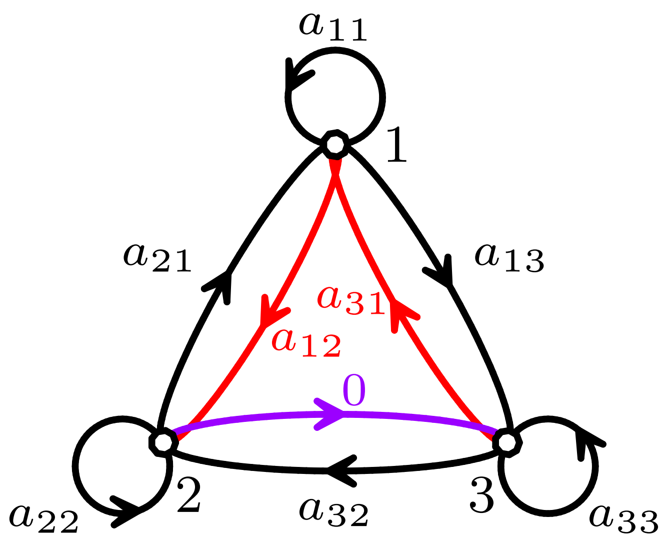


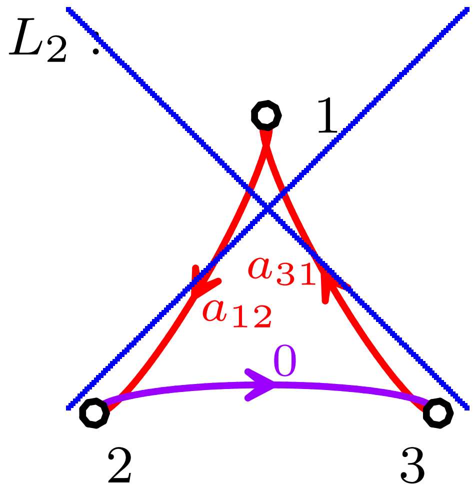
$$\begin{aligned} \det A \quad = \quad & (-1)^{3+3} a_{11} a_{22} a_{33} + (-1)^{3+1} a_{12} a_{31} \cdot 0 + \\ & (-1)^{3+1} a_{21} a_{32} a_{13} + (-1)^{3+2} a_{11} \cdot 0 \cdot a_{32} + \\ & (-1)^{3+2} a_{22} a_{13} a_{31} + (-1)^{3+2} a_{33} a_{12} a_{21} \end{aligned}$$

$$\begin{aligned}\det A \quad = \quad & \color{red}{(-1)^{3+3} a_{11} a_{22} a_{33}} + \qquad 0 \qquad + \\ & (-1)^{3+1} a_{21} a_{32} a_{13} + \qquad 0 \qquad + \\ & (-1)^{3+2} a_{22} a_{13} a_{31} + (-1)^{3+2} a_{33} a_{12} a_{21}\end{aligned}$$

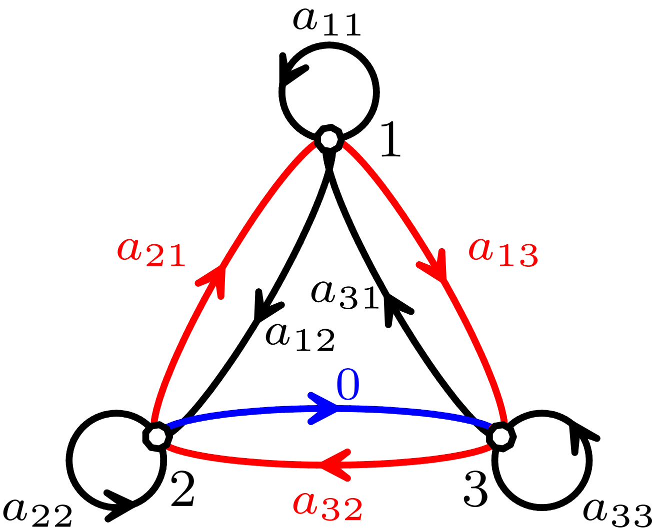


$$\begin{aligned}\det A \quad = \quad & (-1)^{3+3}a_{11}a_{22}a_{33} + \cancel{(-1)^{3+1}a_{12}a_{31}\cdot 0} + \\ & (-1)^{3+1}a_{21}a_{32}a_{13} + \qquad 0 \qquad + \\ & (-1)^{3+2}a_{22}a_{13}a_{31} + (-1)^{3+2}a_{33}a_{12}a_{21}\end{aligned}$$

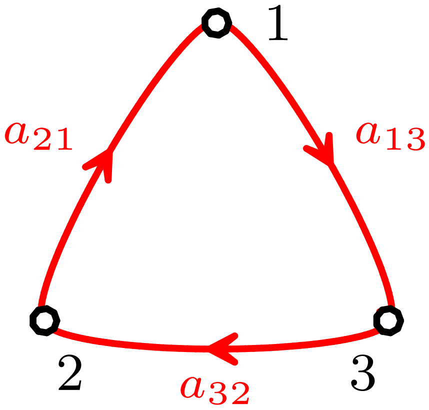




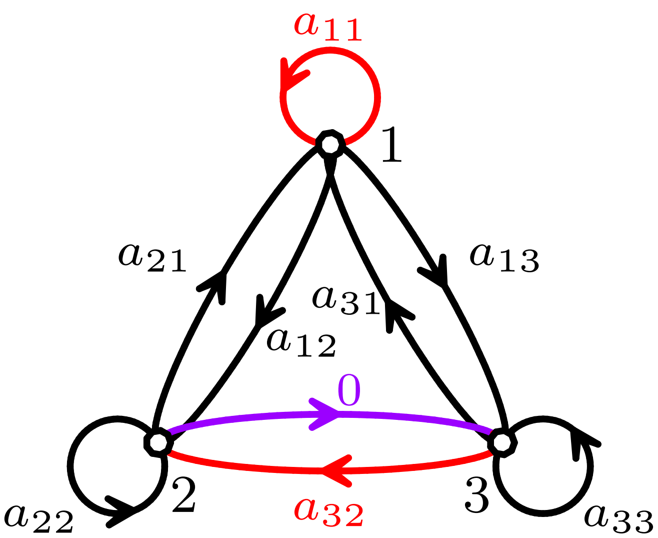
$$
\begin{aligned}
\det A \quad = \quad & (-1)^{3+3} a_{11} a_{22} a_{33} + \qquad 0 \qquad + \\
& {\color{red}(-1)^{3+1} a_{21} a_{32} a_{13}} + \qquad 0 \qquad + \\
& (-1)^{3+2} a_{22} a_{13} a_{31} + (-1)^{3+2} a_{33} a_{12} a_{21}
\end{aligned}
$$

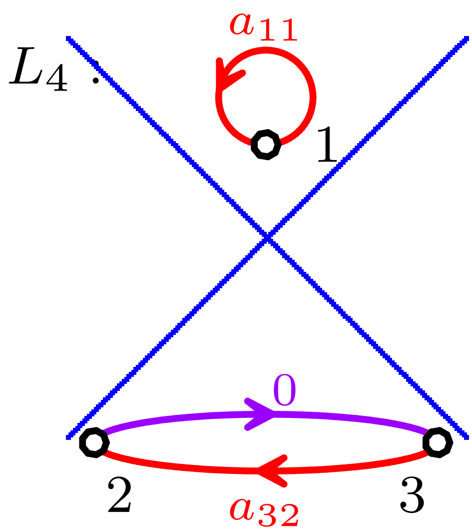


L_3 :

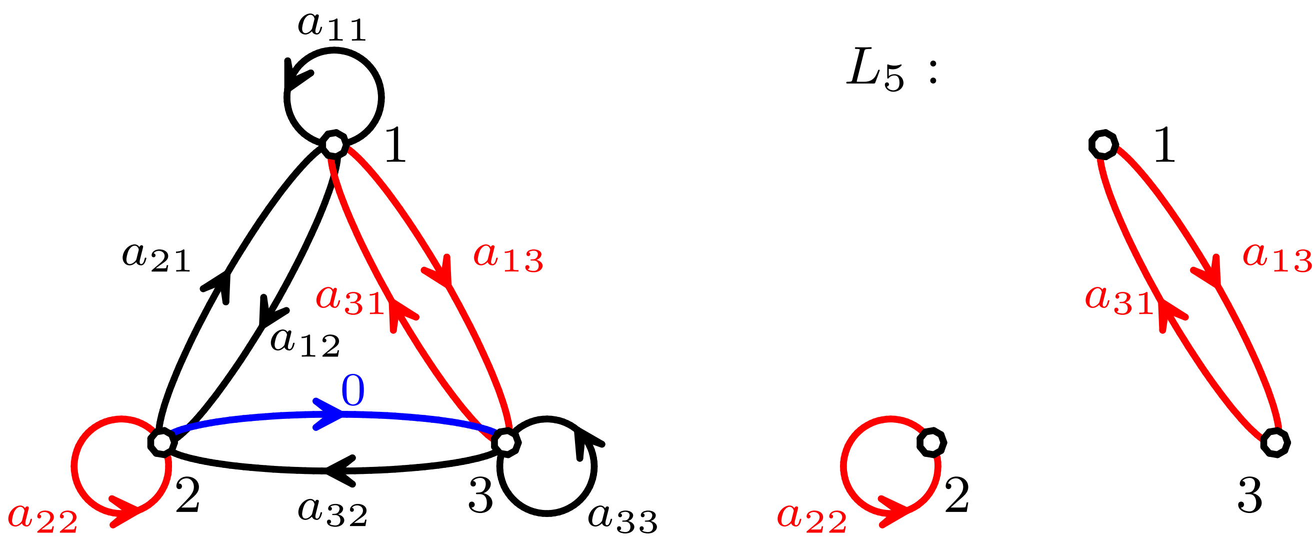


$$
\begin{aligned}
\det A \quad = \quad & (-1)^{3+3}a_{11}a_{22}a_{33} + \qquad 0 \qquad + \\
& (-1)^{3+1}a_{21}a_{32}a_{13} + \cancel{(-1)^{3+2}a_{11}\cdot 0\cdot a_{32}} + \\
& (-1)^{3+2}a_{22}a_{13}a_{31} + (-1)^{3+2}a_{33}a_{12}a_{21}
\end{aligned}
$$

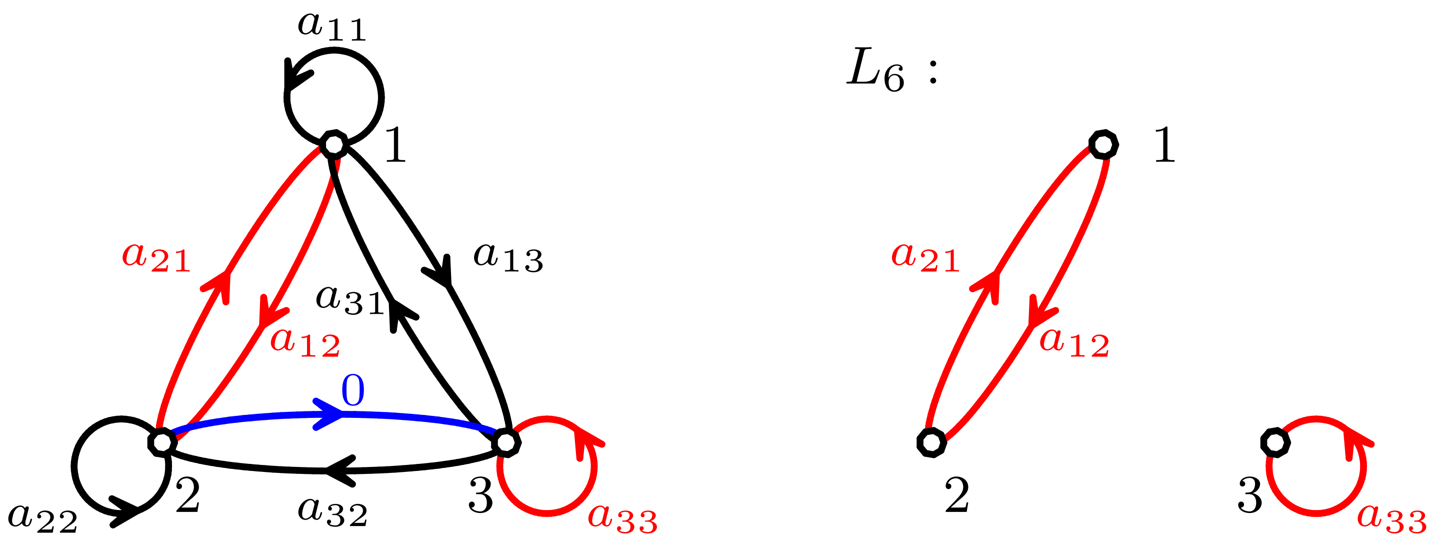




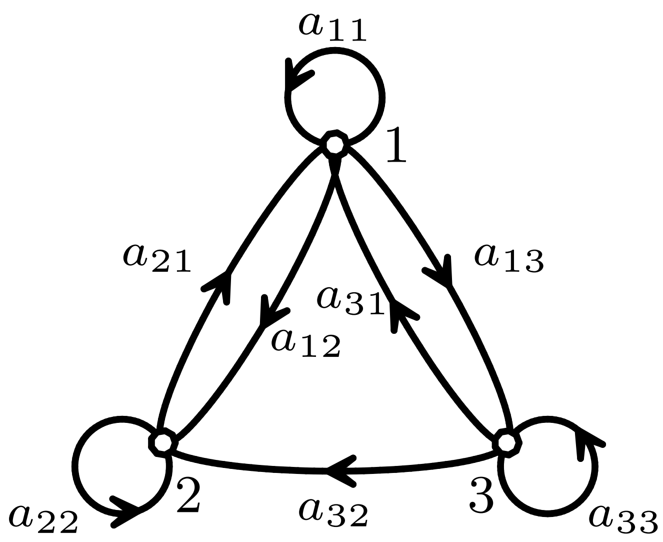
$$
\begin{aligned}
\det A \quad = \quad & (-1)^{3+3} a_{11} a_{22} a_{33} + \qquad 0 \qquad + \\
& (-1)^{3+1} a_{21} a_{32} a_{13} + \qquad 0 \qquad + \\
& \color{red}{(-1)^{3+2} a_{22} a_{13} a_{31}} + (-1)^{3+2} a_{33} a_{12} a_{21}
\end{aligned}
$$



$$
\begin{aligned}
\det A \quad = \quad & (-1)^{3+3} a_{11}a_{22}a_{33} + \qquad 0 \qquad + \\
& (-1)^{3+1} a_{21}a_{32}a_{13} + \qquad 0 \qquad + \\
& (-1)^{3+2} a_{22}a_{13}a_{31} + {\color{red}(-1)^{3+2} a_{33}a_{12}a_{21}}
\end{aligned}
$$

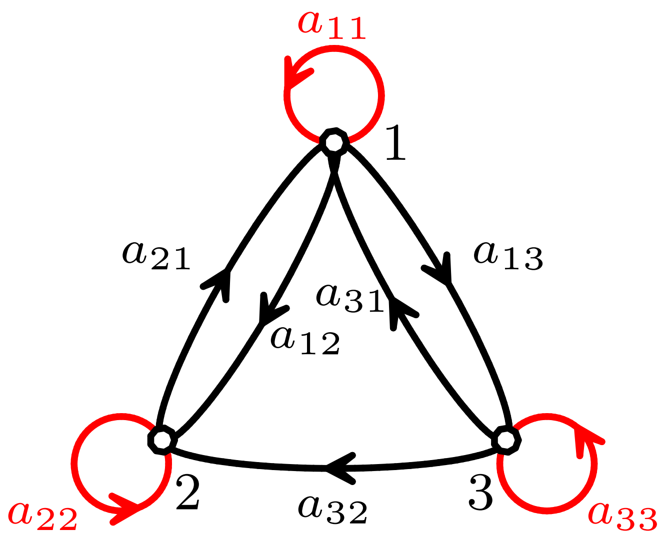


$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

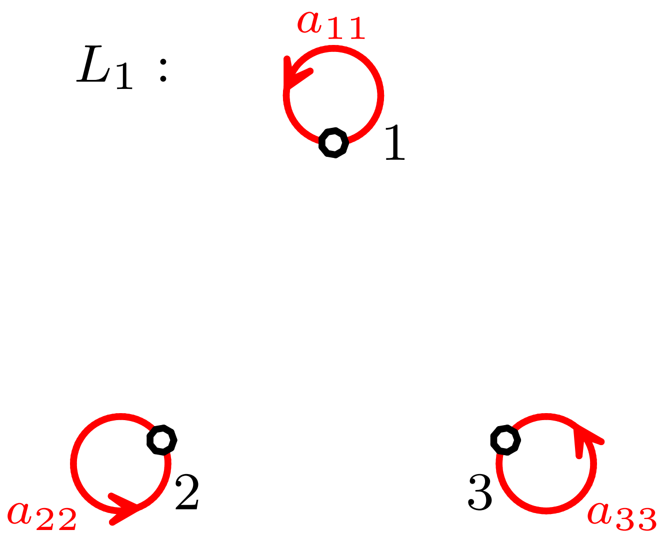


$$\begin{aligned} \det A \quad = \quad & (-1)^{3+3} a_{11} a_{22} a_{33} + \qquad 0 \qquad + \\ & (-1)^{3+1} a_{21} a_{32} a_{13} + \qquad 0 \qquad + \\ & (-1)^{3+2} a_{22} a_{13} a_{31} + (-1)^{3+2} a_{33} a_{12} a_{21} \end{aligned}$$

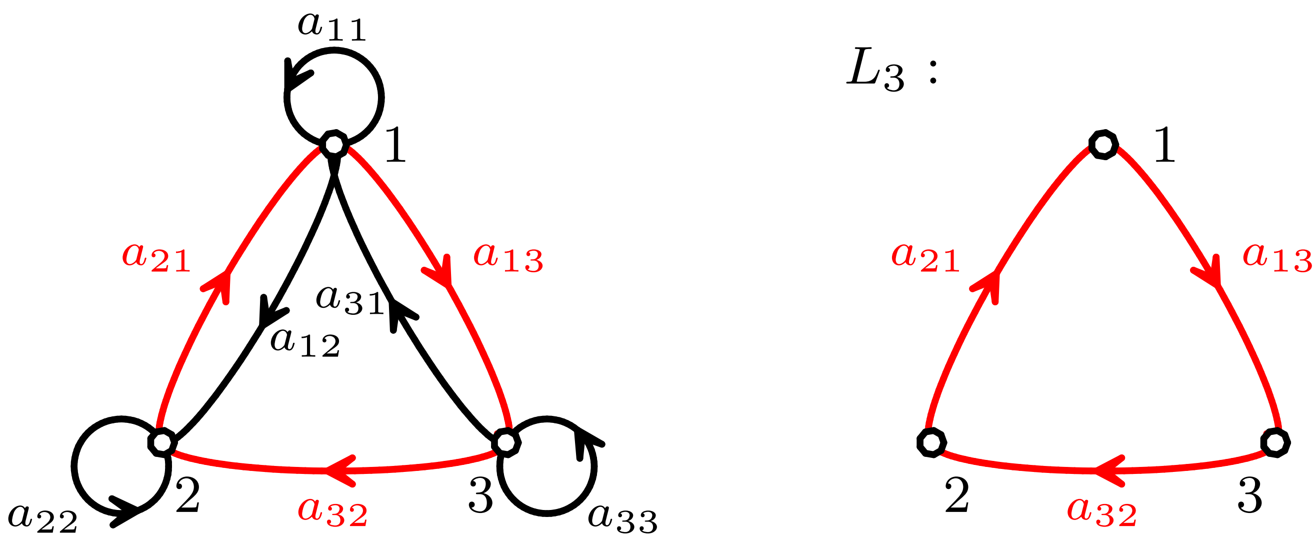
$$
\begin{aligned}
\det A \quad = \quad & \color{red}{(-1)^{3+3} a_{11} a_{22} a_{33}} + \qquad 0 \qquad + \\
& (-1)^{3+1} a_{21} a_{32} a_{13} + \qquad 0 \qquad + \\
& (-1)^{3+2} a_{22} a_{13} a_{31} + (-1)^{3+2} a_{33} a_{12} a_{21}
\end{aligned}
$$



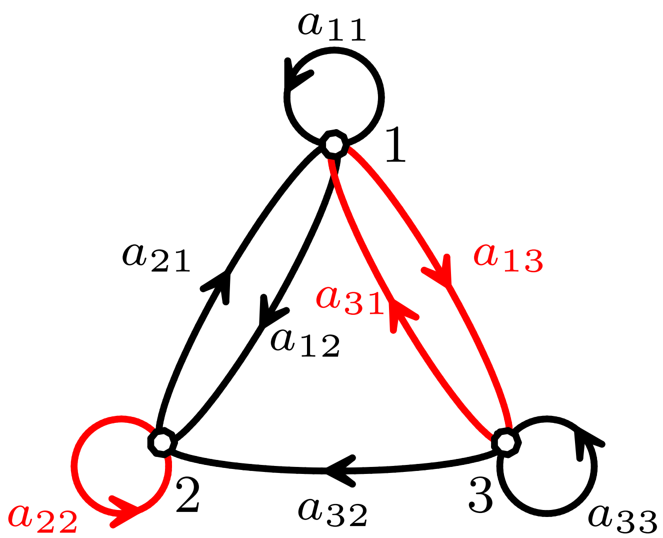
L_1 :



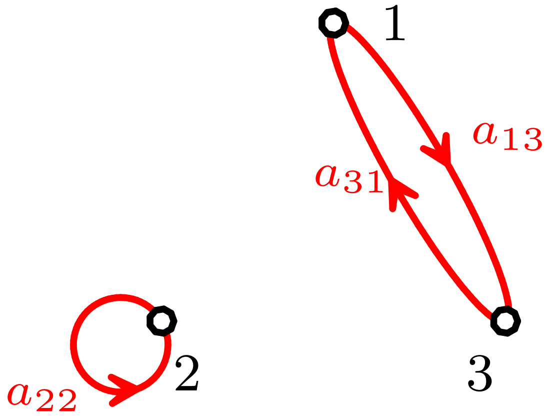
$$\begin{aligned}
\det A \quad = \quad & (-1)^{3+3} a_{11} a_{22} a_{33} + \qquad 0 \qquad + \\
& {\color{red}(-1)^{3+1} a_{21} a_{32} a_{13}} + \qquad 0 \qquad + \\
& (-1)^{3+2} a_{22} a_{13} a_{31} + (-1)^{3+2} a_{33} a_{12} a_{21}
\end{aligned}$$



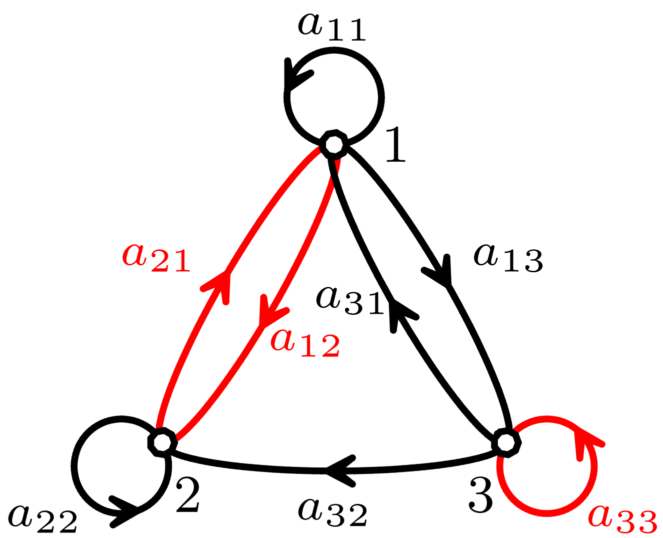
$$
\begin{aligned}
\det A \quad = \quad & (-1)^{3+3}a_{11}a_{22}a_{33} + \qquad 0 \qquad + \\
& (-1)^{3+1}a_{21}a_{32}a_{13} + \qquad 0 \qquad + \\
& \color{red}{(-1)^{3+2}a_{22}a_{13}a_{31}} + (-1)^{3+2}a_{33}a_{12}a_{21}
\end{aligned}
$$



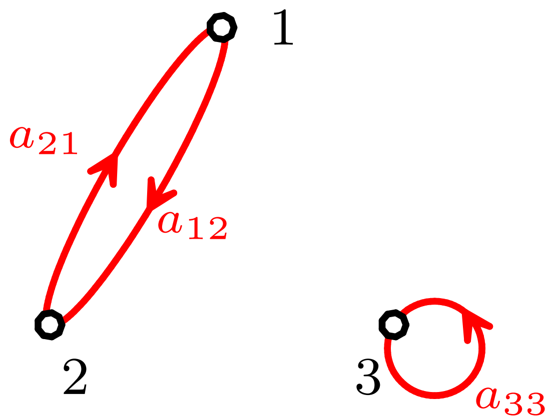
L_5 :



$$
\begin{aligned}
\det A \quad = \quad & (-1)^{3+3} a_{11} a_{22} a_{33} + \qquad 0 \qquad + \\
& (-1)^{3+1} a_{21} a_{32} a_{13} + \qquad 0 \qquad + \\
& (-1)^{3+2} a_{22} a_{13} a_{31} + (-1)^{3+2} a_{33} a_{12} a_{21}
\end{aligned}
$$

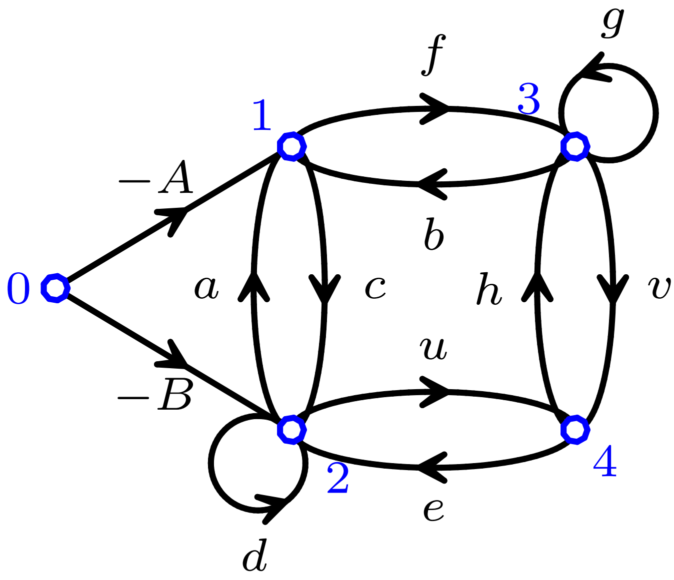


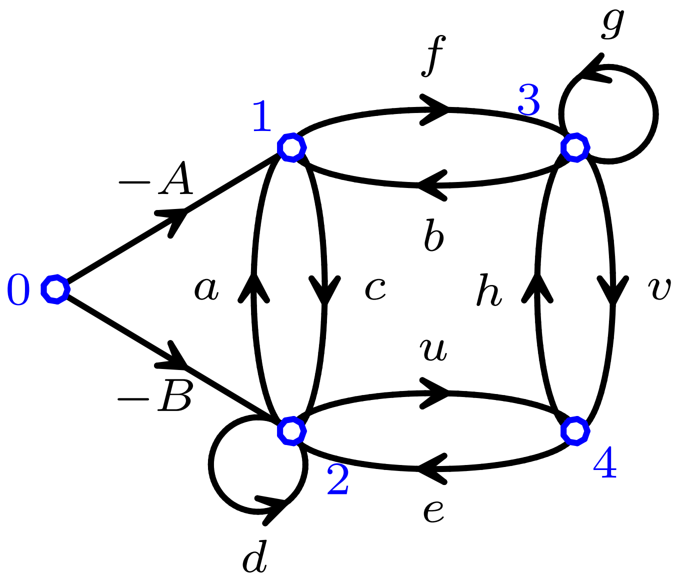
L_6 :



$$
\begin{array}{ccccccccc}
 & & ax_2 & + & bx_3 & & & = & A \\
cx_1 & + & dx_2 & & & + & ex_4 & = & B \\
fx_1 & & & + & gx_3 & + & hx_4 & = & 0 \\
 & & ux_2 & + & vx_3 & & & = & 0
\end{array}
$$

Coates' digraph of the system (flow graph)

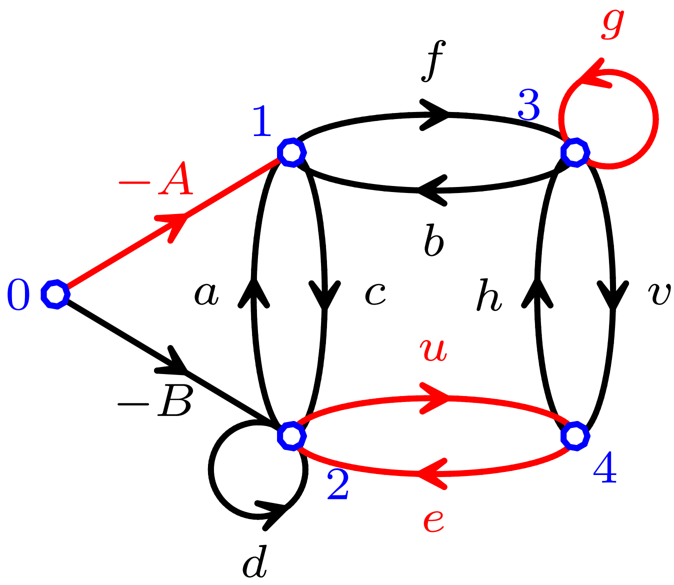




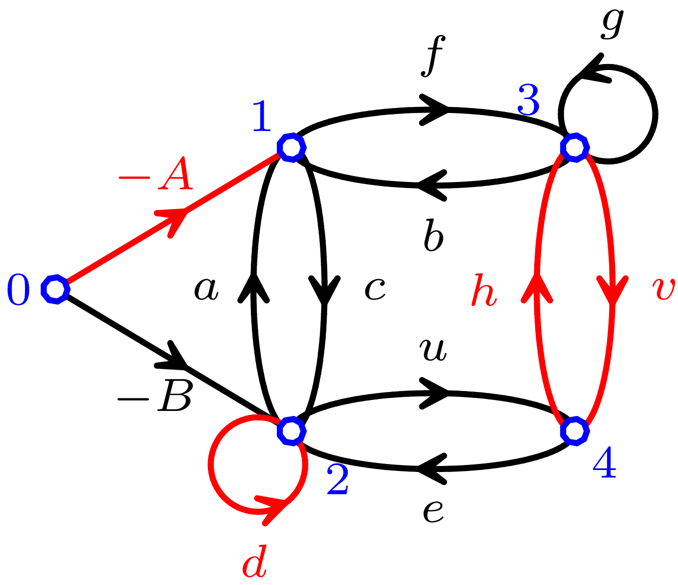
Coates' formula

C.L.Coates, 1959

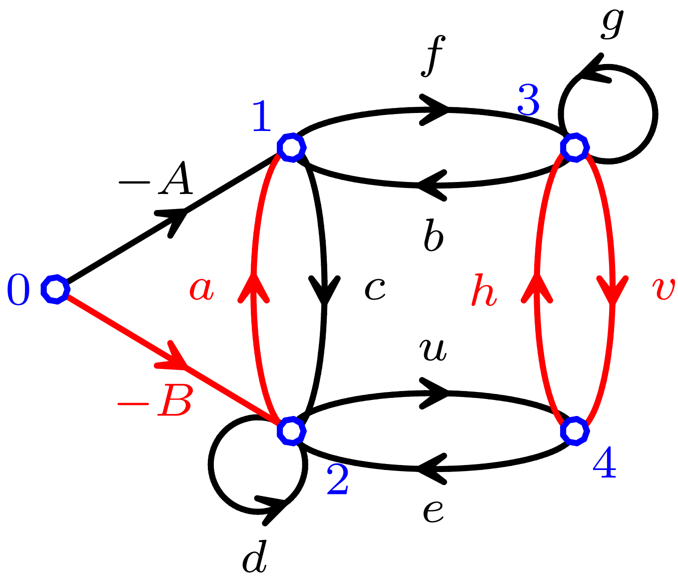
$$x_1 = \frac{-Aeug - Advh + Bavh - Buhb}{acvh + bfeu - bcuh - afve}$$



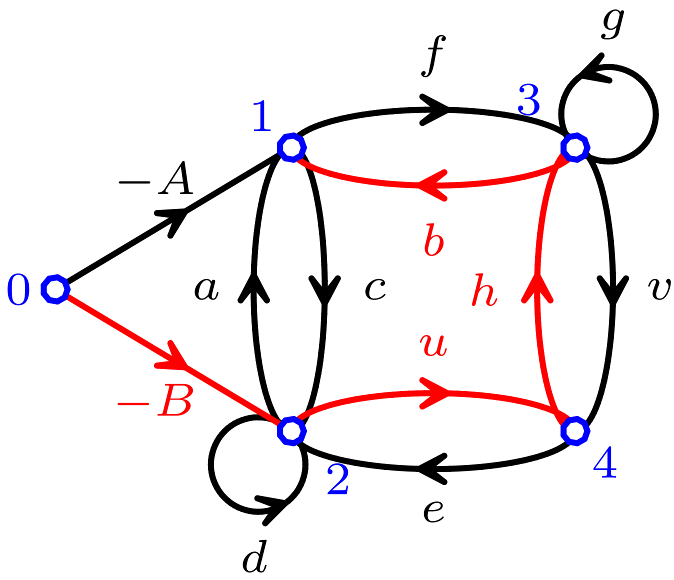
$$x_1 = \frac{-Aeug - Advh + Bavh - Buhb}{acvh + bfeu - bcuh - afve}$$



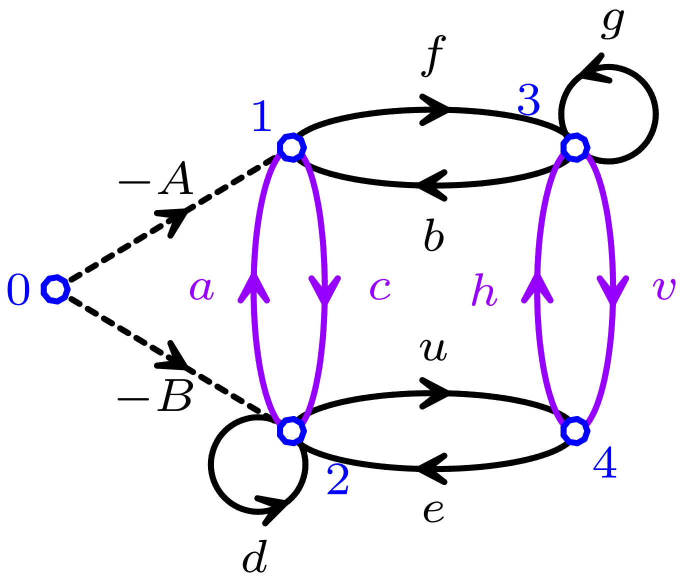
$$x_1 = \frac{-Aeug - Advh + Bavh - Buhb}{acvh + bfeu - bcuh - afve}$$



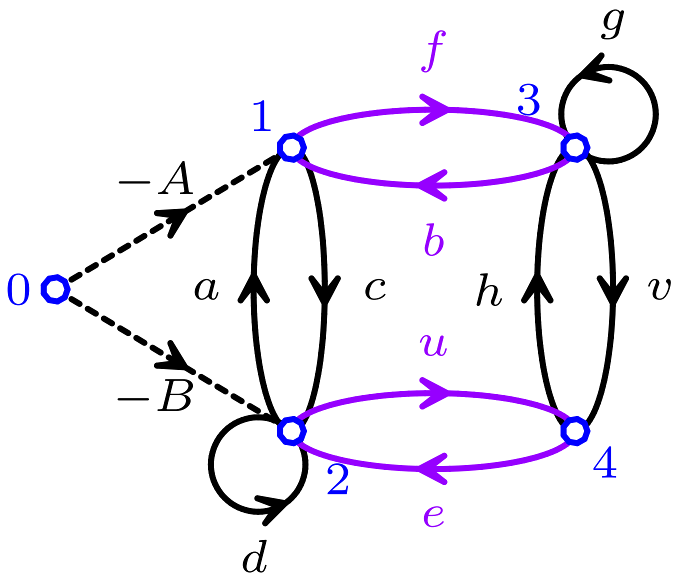
$$x_1 = \frac{-Aeug - Advh + Bavh - Buhb}{acvh + bfeu - bcuh - afve}$$



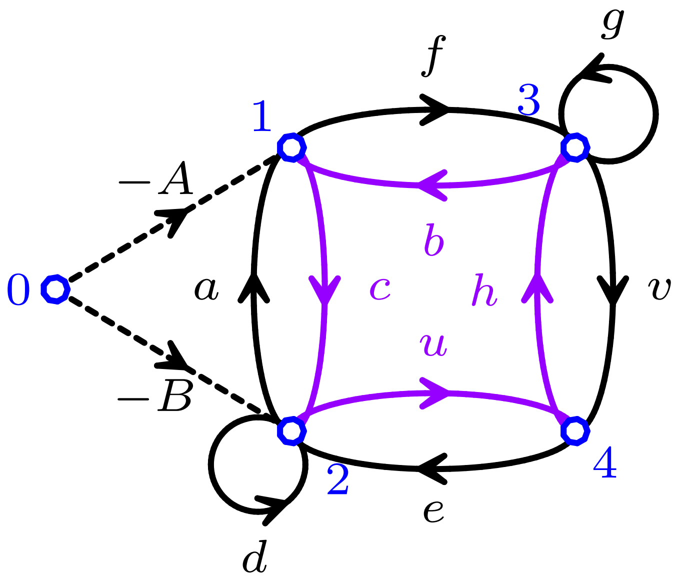
$$x_1 = \frac{-Aeug - Advh + Bavh - Buhb}{acvh + bfeu - bcuh - afve}$$



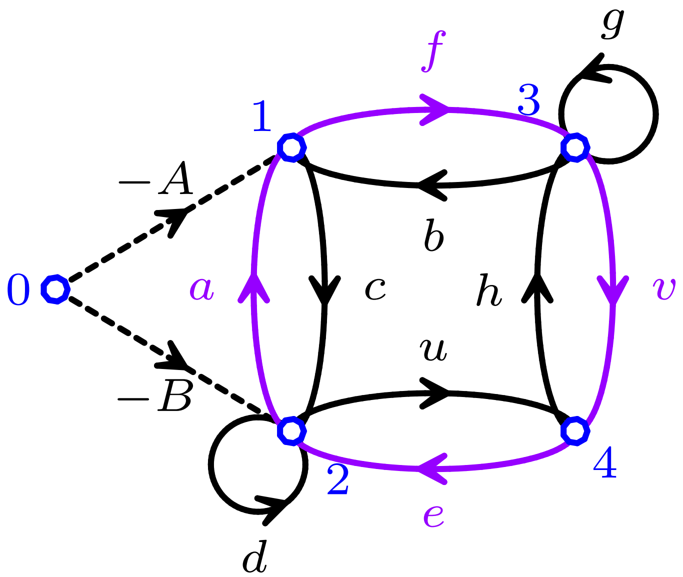
$$x_1 = \frac{-Aeug - Advh + Bavh - Buhb}{acvh + bfeu - bcuh - afve}$$



$$x_1 = \frac{-Aeug - Advh + Bavh - Buhb}{acvh + bfeu - bcuh - afve}$$



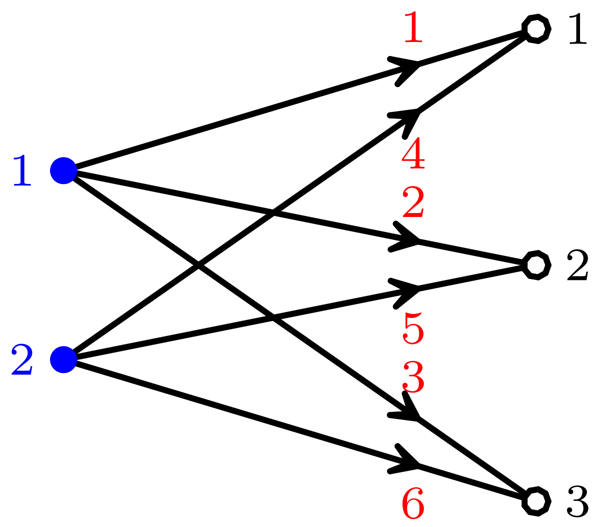
$$x_1 = \frac{-Aeug - Advh + Bavh - Buhb}{acvh + bfeu - bcuh - afve}$$

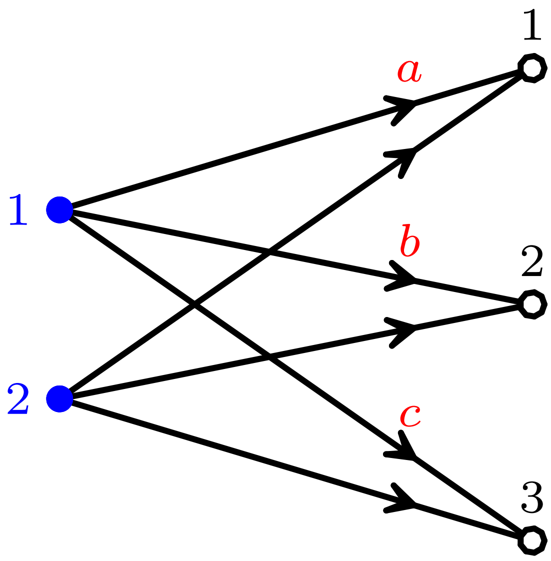


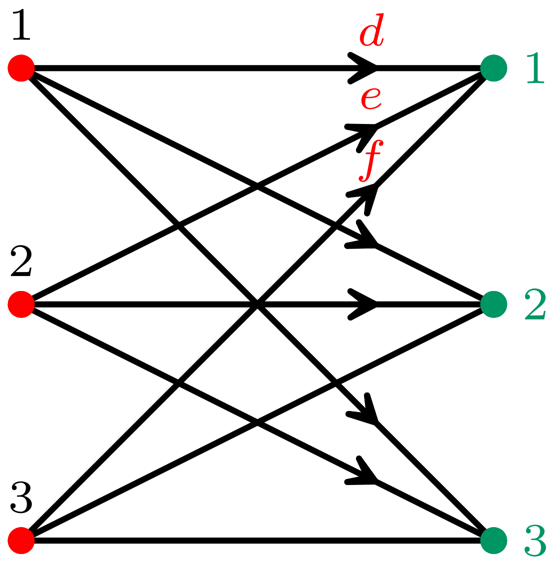
$$x_1 = \frac{-Aeug - Advh + Bavh - Buhb}{acvh + bfeu - bcuh - afve}$$

The *König digraph* of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

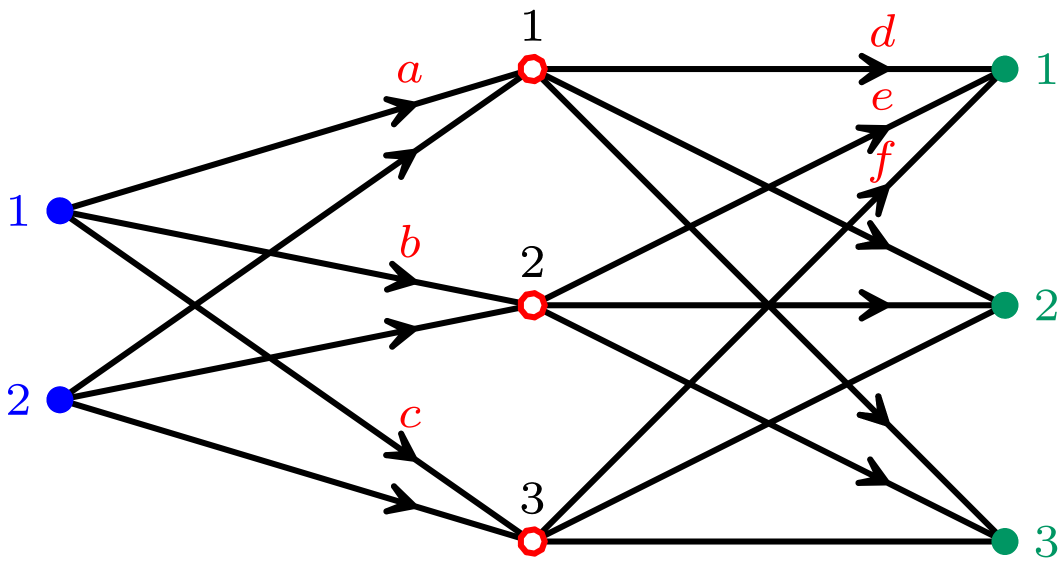






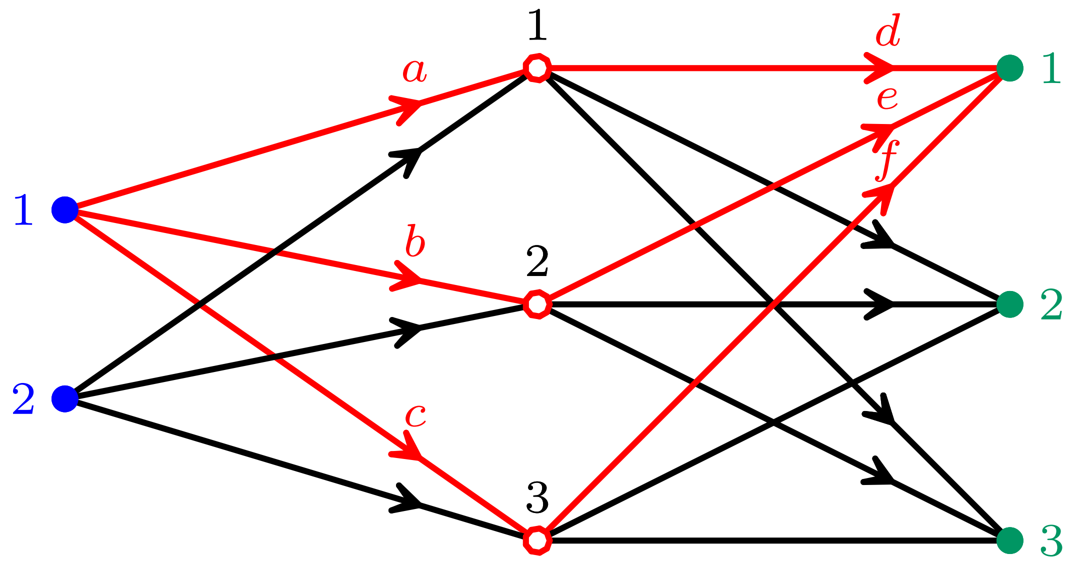
$$\begin{bmatrix} a & b & c \\ \square & \square & \square \end{bmatrix} \qquad \begin{bmatrix} d & \square & \square \\ e & \square & \square \\ f & \square & \square \end{bmatrix}$$

Composition of digraphs



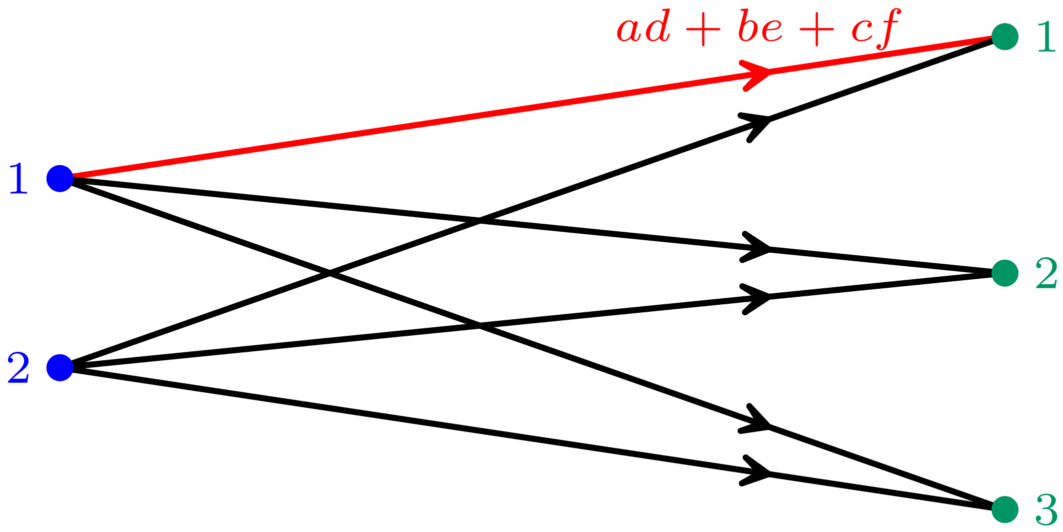
$$\begin{bmatrix} a & b & c \\ \square & \square & \square \end{bmatrix} \cdot \begin{bmatrix} d & \square & \square \\ e & \square & \square \\ f & \square & \square \end{bmatrix}$$

Product of matrices



$$\begin{bmatrix} ad + be + cf & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{bmatrix}$$

Product of matrices



$$\begin{bmatrix} ad+be+cf & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{bmatrix}$$

EXAMPLES: A definition and a theorem

Standard definition. A square matrix A is *reducible* if there is a permutation matrix P such that

$$PAP^T = \begin{bmatrix} X & Y \\ O & Z \end{bmatrix} \tag{1}$$

where X and Z are square matrices of order at least 1. The matrix A is *irreducible* provided the form (1) cannot be achieved for any permutation matrix P.

EXAMPLES: A definition and a theorem

Standard definition. A square matrix A is *reducible* if there is a permutation matrix P such that

(1) $$PAP^T = \begin{bmatrix} X & Y \\ O & Z \end{bmatrix}$$

where X and Z are square matrices of order at least 1. The matrix A is *irreducible* provided the form (1) cannot be achieved for any permutation matrix P.

New definition. A square matrix A is *irreducible* if its digraph $D(A)$ is strongly connected; otherwise, A is *reducible*.

Theorem. Let A be a square matrix of order n. Then A is nilpotent if the corresponding digraph $D(A)$ does not have any cycles; in this case, $A^n = O$. A nonnegative square matrix A is nilpotent if and only if the corresponding digraph $D(A)$ does not have any cycles.

This book

- Places combinatorial and graph-theoretical tools at the forefront of the development of matrix theory;

This book

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Thank you for your attention.