Conormal Lagrangian Floer homology for open subsets and PSS isomorphism

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Morse homology

- M closed manifold
- $f : M \to \mathbb{R}$ Morse function
- $\mathrm{Crit}(f) \rightsquigarrow$ generators
- $\sharp\{\gamma \mid \dot{\gamma} = -\nabla f\} \rightsquigarrow$ boundary operator $\partial_M \rightsquigarrow HM(f)$
- canonical isomorphism $HM(f^\alpha) \cong HM(f^\beta)$
- $HM(f) \cong H^{\mathrm{sing}}(M)$

Lagrangian Floer homology

- (P, ω) symplectic manifold
- $L_0, L_1 \subset P$ Lagrangian
- $L_0 \cap L_1 \rightsquigarrow$ generators
- $\sharp\{$holomorphic strips u, boundary conditions: $u(s,j) \in L_j\} \rightsquigarrow$ boundary operator ∂_F
- Floer homology $HF(L_0, L_1)$
- $P = T^*M$, $L_0 = O_M$, $L_1 = \phi_H^1(O_M)$, there is a canonical isomorphism $HF(O_M, \phi^1_{H^\alpha}(O_M)) \cong HF(O_M, \phi^1_{H^\beta}(O_M))$
- Floer: for special choice of f and H, $HF(O_M, \phi_H^1(O_M)) \cong HM(f)$
- this isomorphism is not functorial w.r.t. canonical isomorphisms in Morse and Floer theory

PSS isomorphism

- Piunikhin – Salamon – Schwarz (1996), K. – Milinković (2005), Albers (2008), Đuretić (2014)
- PSS is defined by a number of combined objects (γ, u), where γ is a negative gradient trajectory (from ∂_M) and u is a holomorphic strip (from ∂_F)
- PSS is functorial:

$$\begin{array}{ccc} HM(f_\alpha) & \xrightarrow{S_{\alpha\beta}} & HM(f_\beta) \\ \downarrow{\scriptstyle PSS_\alpha} & & \downarrow{\scriptstyle PSS_\beta} \\ HF(O_M, \phi^1_{H_\alpha}(O_M)) & \xrightarrow{T_{\alpha\beta}} & HF(O_M, \phi^1_{H_\beta}(O_M)) \end{array}$$

commutes

Floer homology for submanifolds

- Pozniak (1994)
- $N \subset M$ closed sumbanifold; conormal bundle:

$$\nu^* N := \{\alpha \in T^*M|_N \mid \alpha_{TN} = 0\}$$

  is always a Lagrangian submanifold
- Floer homology $HF(\nu^* N, \phi_H^1(O_M))$ is isomorphic to $H_{\mathrm{sing}}(N)$
- also not functorial
- Đuretić (2014): construction of PSS type isomorphism $HM(f, N) \xrightarrow{\cong} HF(\nu^* N, \phi_H^1(O_M))$ which is functorial

Floer homology for open subsets

- Kasturirangan, Oh (2001)
- $U \subset M$ open, ∂U smooth
- $\nu^*_-(\partial U) := \{\alpha \in \nu^*(\partial U) \mid \alpha(\mathbf{n}) \leq 0\}$, for $\mathbf{n}$ outward normal to ∂U
- $\nu^*\overline{U} := O_U \cup \nu^*_-(\partial U)$ - conormal to $\overline{U}$
- $\nu^*\overline{U}$ is not a smooth manifold
- there are approximations Υ of $\nu^*\overline{U}$, Υ smooth Lagrangian
- $HF(\phi^1_H(O_U), \Upsilon)$
- there is a partial order on $\{\Upsilon^s\}$, and homomorphisms $F_{ab} : HF(\phi^1_H(O_U), \Upsilon^a) \to HF(\phi^1_H(O_U), \Upsilon^b)$ such that $F_{bc} \circ F_{ab} = F_{ac}$, for $\Upsilon^a \leq \Upsilon^b \leq \Upsilon^c$
- Floer homology as a direct limit
  $HF(U) := \underline{\lim}_s HF(\phi^1_H(O_U), \Upsilon^s)$
- $HF(U) \cong HM(f, U) \cong H^{\text{sing}}(M)$ for special f

PSS for open subset

- K. – Milinković – Nikolić
- f such that $\mathrm{Crit}(t) \cap \partial U = \emptyset$, ∇f points outward ∂U (to avoid some analytical troubles)
- first step: PSS for approximations

$$PSS^{\Upsilon} : HM(f, U) \to HF(\phi_H^1(O_U), \Upsilon^s)$$

- second: PSS commutes with F_{ab} that defines direct limit $\rightsquigarrow$ $PSS : HM(f, U) \to HF(U)$
- functoriality: all PSS^{Υ} are functorial

Poincaré duality - Morse case

- cannot define the other way around PSS via number of (u, γ) for same f
- for $-f$ instead of f $PSS : HF(U) \to HM(f, U)$ is well defined
- PD for Morse:

$$H_k^{\text{sing}}(U) \cong HM_k(f, U) \cong HM_{n-k}(-f, U) \cong H_{n-k}^{\text{sing}}(U, \partial U)$$

for f as above

Poincaré duality - Floer case

- instead of $\nu_-^*(\partial U)$ consider

$$\nu_+^*(\partial U) := \{\alpha \in \nu^*(\partial U) \mid \alpha(\mathbf{n}) \geq 0\}$$

- $\zeta : x = (q,p) \mapsto \overline{x} := (q,-p)$
- $\zeta(\nu_-^* \partial U) = \nu_-^* \overline{U}$
- Υ - approximation of $\nu_-^* \overline{U} \Rightarrow \overline{\Upsilon} := \zeta(\Upsilon)$ approximation of $\nu_+^* \overline{U}$
- $\overline{H}(x,t) := H(\zeta(x),t) \rightsquigarrow \phi_H^1(O_U) \cap \Upsilon \cong \phi_{\overline{H}}^1(O_U) \cap \overline{\Upsilon}$
- $\overline{J} := \zeta^* J \rightsquigarrow$ bijection of the spaces of holomorphic discs defining the boundary operation
- $\zeta_* : HF_k(\phi_H^1(O_U), \Upsilon : J) \xrightarrow{\cong} HF_{n-k}(\phi_{\overline{H}}^1(O_U), \overline{\Upsilon} : \overline{J})$
- $\mathrm{PD}_{Floer} = \zeta_*$

PSS is isomorphism

If Ψ is defined by a number of (u, γ), then

$$\begin{array}{ccc} HM_k(f, U) & \xrightarrow{PSS} & HF_k(\phi^1_H(O_U), \Upsilon) \\ \text{PD}_{Morse} \downarrow \cong & & \text{PD}_{Floer} \downarrow \cong \\ HM_{n-k}(-f, U) & \xleftarrow{\Psi} & HF_{n-k}(\phi^1_{\overline{H}}(O_U), \overline{\Upsilon}) \end{array}$$

commutes

$\Rightarrow$ PSS is isomorphism

Isomorphism for a direct limit

If PSS, PD_{Morse}, PD_{Floer}, Ψ all commute with homomorphisms defining a direct limits, then

$$\begin{array}{ccc} HM_k(f,U) & \xrightarrow{PSS} & HF_k(U) \\ \mathrm{PD}_{Morse}\downarrow\cong & & \mathrm{PD}_{Floer}\downarrow\cong \\ HM_{n-k}(-f,U) & \xleftarrow{\Psi} & HF_{n-k}(U,\overline{\Upsilon}) \end{array}$$

commutes