Compact and "compact" operators on the standard Hilbert module over a W^*-algebra

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Algebras

C^*-algebra A Banach algebra with an involution such that $||a^*a|| = ||a||^2$.

Any C^* algebra has a representation as a subalgebra of $B(H)$ for some Hilbert space H – Gelfand-Naimark theorem.

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Examples

- $C(K)$ is a C^*-algebra, but not W^*.
- $B(H)$ is a W^*-algebra. $B(H)_* \cong \mathfrak{S}_1$.
- $L^\infty(X;\mu)$ is a W^*-algebra. $L^\infty(X;\mu)_* \cong L^1(X;\mu)$.

Hilbert C^* modules

Hilbert C^*-module A right module M over A with an A-valued inner product such that

1. $\langle a, a\rangle \geq 0$, $\langle a, a\rangle = 0 \Leftrightarrow a = 0$;
2. $\langle b, a\rangle = \langle a, b\rangle^*$;
3. $\langle a, b_1\lambda_1 + b_2\lambda_2\rangle = \langle a, b_1\rangle \lambda_1 + \langle a, b_2\rangle \lambda_2$.

Here, a, b, $b_j \in M$, $\lambda_j \in A$.
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Standard Hilbert module

$l^2(A) = \{x = (\xi_1, \xi_2, \dots) \mid \xi_j \in A, \sum_{j=1}^{+\infty} \xi_j^* \xi_j \text{ conv. in } ||\cdot||\}$.

Standard Hilbert module over a unital algebra has a (Riesz) basis $e_j = (0, 0, \dots, 1, 0, \dots)$, 1 is the unit of A placed on j-th entry.

"Compact" operators

"Compact" operators on a module M

Closed linear span of the operators $\Theta_{y,z} : M \to M$, $\Theta_{y,z}(x) = z \langle y, x \rangle$.

Such operators need not to map bounded sets into relatively compact. Hence the quotation marks.

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Problem

Find a topology on $l^2(A)$ such that "compact" operators map bounded into totally bounded sets.

If possible, prove the converse, if A maps bounded into totally bounded sets then A is "compact".

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Result

The first problem is solved. The second partially.

Locally convex spaces

A locally convex space

Determined by a family of seminorms p_i, $i \in I$.
Seminorms gives rise to the family of semimetrics $d_i(x, y) = p_i(x - y)$.
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A set is totally bounded if it is totally bounded in all d_i.
A net is a Cauchy net if it is Cauchy net in all d_i.
A space is complete if all Cauchy nets converge.
Relatively compact $\Rightarrow$ totally bounded.
The converse is true provided the space is complete.

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In further

M is a Hilbert module over a W^*-algebra. From now to the end.

Paschke-Frank topologies

PF topologies

Weak and strong topology:

- weak τ_1 generated by functionals of the form $M \ni x \mapsto \varphi(\langle y, x\rangle)$, $y \in M$, φ a normal state. Seminorms $|\varphi(\langle y, x\rangle)|$.
- strong τ_2 generated by seminorms $\varphi(\langle x, x\rangle)^{1/2}$, φ a normal state.

Always $\tau_1 \subset \tau_2$.

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Normal states Positive functionals $\varphi \in A_*$, $||\varphi|| = 1$. Then $\varphi(1) = 1$.

To continue, we need a definition of a dual module.

Selfdual module A module M over A such that all A-linear maps from M to A are of the form $x \mapsto \langle y, x\rangle$ for some $y \in M$.

Otherwise, the space of A linear maps forms another module M' – the dual module. M' is always selfdual.

Properties of PF topologies

Weak PF topology

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τ_1 – exactly weak-$*$ topology.

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The following is equivalent:

- M is selfdual;
- The unit ball in M is complete in τ_1;
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PF are not suitable

τ_1 is too weak – the unit ball in $l^2(A)$ is compact.
τ_2 is too strong – the unit ball in A^n is not compact.

Here is the right topology – τ!

Definition $M = l^2(A)$ – the standard Hilbert module. Topology τ generated by seminorms

$$p_{\varphi,y}(x) = \sqrt{\sum_{j=1}^{+\infty} |\varphi(\eta_j^* \xi_j)|^2},$$

where φ is a normal state and $y = (\eta_1, \eta_2, \dots)$ satisfies

$$\sup_j \varphi(\eta_j^* \eta_j) < +\infty. \tag{1}$$

Note that y need not to $\in l^2(A)$. However, for any $y = (\eta_1, \eta_2, \dots)$, the sequence $\eta_j / \varphi(\eta_j^* \eta_j)^{1/2}$ fulfills (1).

Properties of τ

Properties

1. $\tau_1 \subset \tau \subset \tau_2$;
2. The unit ball in $l^2(A)$ is not complete in all τ_1, τ, τ_2 ($l^2(A)$ is not selfdual);
3. Restricted to A^n (forget all after n-th entry) $\tau_1 = \tau$;
4. The unit ball in A^n is compact in τ, hence totally bounded (A^n is selfdual);
5. The unit ball in $l^2(A)$ is not totally bounded in τ.

Property 1

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Proof

$\tau_1 \subset \tau$ $y = (\eta_1, \eta_2, \dots) \in l^2(\mathcal{A}) \Rightarrow \zeta_j = \eta_j / \varphi(\eta_j^* \eta_j)^{1/2}$ fulfils (1). Hence

$$|\varphi(\langle y, x \rangle)| = \left| \varphi \left(\sum_{j=1}^{+\infty} \eta_j^* \xi_j \right) \right| = \left| \sum_{j=1}^{+\infty} \varphi(\eta_j^* \eta_j)^{\frac{1}{2}} \varphi(\zeta_j^* \xi_j) \right| \leq$$

$$\leq \left(\sum_{j=1}^{+\infty} \varphi(\eta_j^* \eta_j) \right)^{\frac{1}{2}} \left(\sum_{j=1}^{+\infty} |\varphi(\zeta_j^* \xi_j)|^2 \right)^{\frac{1}{2}} =$$

$$= \varphi(\langle y, y \rangle)^{\frac{1}{2}} p_{\varphi, z}(x).$$

Property 1 - continuation

$\tau \subset \tau_2$ $(\xi, \eta) \mapsto \varphi(\xi^* \eta)$ – a semi inner product. Hence $|\varphi(\xi^* \eta)| \leq \varphi(\xi^* \xi)^{\frac{1}{2}} \varphi(\eta^* \eta)^{\frac{1}{2}}$.

$$p_{\varphi,y}(x)^2 = \sum_{j=1}^{+\infty} |\varphi(\eta_j^* \xi_j)|^2 \leq \sum_{j=1}^{+\infty} \varphi(\xi_j^* \xi_j) \varphi(\eta_j^* \eta_j) \leq$$

$$\leq \sum_{j=1}^{+\infty} \varphi(\xi_j^* \xi_j) = \varphi(\langle x, x \rangle).$$

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Remark These proofs works also for $l^2(A)' = \{x = (\xi_n)_{n\geq 1} \mid \sup_n || \sum_1^n \xi_j^* \xi_j || < +\infty\}$.

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Proof

Unit ball in $l^2(A)'$ is complete

$l^2(A)' \ni x^\alpha$ Cauchy net $\Rightarrow \xi_k^\alpha$ C. net in A. (Choose $\eta_k = 1$, $\eta_j = 0$ for $j \neq k$.) Hence $\xi_k^\alpha \stackrel{w*}{\to} \xi_k$, and $\sum_{j=1}^k |\varphi(\eta_j^* \xi_j^\alpha)|^2 \to \sum_{j=1}^k |\varphi(\eta_j^* \xi_j)|^2$.

Let $\eta_j = \xi_j / \varphi(\xi_j^* \xi_j)^{\frac{1}{2}}$. We get

$$\sum_{j=1}^k \varphi(\xi_j^* \xi_j) = \sum_{j=1}^k |\varphi(\eta_j^* \xi_j)|^2 = \lim_\alpha \sum_{j=1}^k |\varphi(\eta_j^* \xi_j^\alpha)|^2 \leq$$
$$\leq ||x|| \leq 1.$$

Take $\lim_{k \to +\infty}$ to conclude $x = (\xi_1, \xi_2, \dots) \in l^2(\mathcal{A})'$.

Property 2 - continuation

Finally

$$\sum_{j=1}^{k} |\varphi(\eta_j^* \xi_j^\alpha) - \varphi(\eta_j^* \xi_j^\beta)|^2 \leq \sum_{j=1}^{+\infty} |\varphi(\eta_j^* \xi_j^\alpha) - \varphi(\eta_j^* \xi_j^\beta)|^2 < \varepsilon,$$

take the limit over β and limit as $k \to +\infty$.

Property 2 - continuation

The ball in $l^2(A)$ is dense in the ball in $l^2(A)'$

Finally

$$\sum_{j=1}^{k} |\varphi(\eta_j^* \xi_j^\alpha) - \varphi(\eta_j^* \xi_j^\beta)|^2 \leq \sum_{j=1}^{+\infty} |\varphi(\eta_j^* \xi_j^\alpha) - \varphi(\eta_j^* \xi_j^\beta)|^2 < \varepsilon,$$

take the limit over β and limit as $k \to +\infty$.

$(\xi_1, \xi_2, \dots, \xi_n, 0, 0, \dots) = x_n \to x = (\xi_1, \dots) \in l^2(A)'$.
Indeed, by normality of φ we have

$$p_{\varphi,y}(x - x_n)^2 \leq \varphi(\langle x - x_n, x - x_n \rangle) = \varphi\left(\sum_{j=n}^{+\infty} \xi_j^* \xi_j\right) \to 0,$$

as $n \to +\infty$.

Properties 3 & 4

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$$p_{\varphi,y}(x) = \left(\sum_{j=1}^{n} |\varphi(\eta_j^* \xi_j)|^2\right)^{\frac{1}{2}} \leq$$

$$\leq \sum_{j=1}^{n} |\varphi(\eta_j^* \xi_j)| = \sum_{j=1}^{n} |\varphi(\langle z_j, x\rangle)|,$$

where $z_j = (0, \ldots, 0, \eta_j, 0, \ldots, 0)$.

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Follows easily from the previous.

Property 5

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There is a totally discrete sequence.

Choose $\eta_j = 1$ for all j and φ arbitrary. Then $p_{\varphi,y}(e_n - e_m) = \sqrt{2}$.

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Proposition T is "compact" $\Rightarrow$ A is compact (i.e. maps bounded into totally bounded sets).

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Proof Observe the following three facts:

1. Projections $P_n : l^2(A) \to l^2(A)$, $P_n(\xi_1, \dots, \xi_n, \xi_{n+1}, \dots) = (\xi_1, \dots, \xi_n, 0, \dots)$ make an approximate identity in the algebra of "compact" operators;
2. The property of being compact is A linear;
3. The property of being compact is stable under limits, either uniform, or in τ.

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The first of them is well known. The last two are easy to derive.

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Appr. by basis Otherwise, if $z = (\zeta_1, \zeta_2, \dots)$. Then $z = \sum_{j=1}^{+\infty} e_j\zeta_j$ (conv. in the norm). Since $||\Theta_{y,z} - \Theta_{y,z'}|| \leq ||y||\,||z - z'||$, we have

$$\Theta_{y,z} = \lim_{n\to+\infty} \sum_{j=1}^{n} \Theta_{y,e_j\zeta_j}.$$

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For T not "compact" construct a totally discrete sequence in the image of the unit ball. The proof is highly technical (many indices, ε's etc.) – hence omitted. The following Lemma plays the key roll.

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The Lemma Let $a_n \in B(H)$ be a sequence of positive operators, such that $||a_n|| > \delta$. Then there is a normal state φ, and unitaries υ_n, ν_n such that $\varphi(\upsilon_n^* a_n \nu_n) > \delta$.

A counterexample

For A commutative

Let $p_j \in A$ be mutually orthogonal nontrivial projections. Then $T : l^2(A) \to l^2(A)$,

$$Tx = T(\xi_1, \xi_2, \dots) = (p_1\xi_1, p_2\xi_2, \dots)$$

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Proof (outline)

To see T is not "compact" observe that $||T - P_n T|| = 1$ contradicting P_n is an approximate identity.
To see T is compact, note for any seminorm $p(T - P_n T) \to 0$. This preserves compactness.

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Also the sequence p_n contradicts the Lemma.

Problems for further work

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Problem 3 Extend the presented results to any module over A (not only for $l^2(A)$), i.e. make seminorms independent of coordinates. It might be difficult.

Thanks for your attention

Complete proofs at https://arxiv.org/abs/1610.06956

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To make it easier to remember:

1610	Henri IV of France assassinated by Ravaillac
069	it is easy to remember. I guess?!
56	due to previous students often get 5 and 6