Holomorphic discs in complex manifolds

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Abstract

I will present a method of constructing proper holomorphic maps from the unit disc in $\mathbb{C}$ into complex manifolds. For the local step an approximate solutions to a certain Riemann-Hilbert problem is used and local corrections are glued using holomorphic sprays.

These two methods also apply in Poletsky theory, i.e. in the study of plurisubharmonic extremal functions as envelopes of dics functionals, and recently, in minimal surface theory.

Notation and definitions

- Let $\mathbb{D}$ denote the open unit disc in $\mathbb{C}$.
- A *holomorphic disc f in X* is a holomorphic map $f\colon \mathbb{D} \to X$, where X may be a complex Euclidean space $\mathbb{C}^n$, a complex manifold or a complex space with singularities. We also call f an *analytic disc*. If the map f extends continuously up to the boundary, we call f a *closed analytic disc*. The point $f(0)$ is called *the center* of the analytic disc f.
- A holomorphic disc $f\colon \mathbb{D} \to X$ is *proper* if for any compact set $K \subset X$ the set $f^{-1}(K)$ is compact in $\mathbb{D}$.
- The image $f(\mathbb{D})$ is a complex subvariety of X.

Previous results

Question: Let X and a point $p \in X$ are given. Are there any proper holomorphic discs $f\colon \mathbb{D} \to X$ such that $f(0) = p$?

- Yes, for any point p in a convex domain $X \subset \mathbb{C}^n$, $X \neq \mathbb{C}^n$.
- Forstnerič, Globevnik (1992): Yes, for any point p in a strictly pseudoconvex domain $X \subset \mathbb{C}^n$, $n \geq 1$. No, in general. There exists a smoothly bounded domain $X \subset \mathbb{C}^n$ and a point $p \in X$ such that there are no proper holomorpic discs through p.
- Dor (1996) There exists a bounded domain $X \subset \mathbb{C}^n$ with no proper holomorpic discs.
- Globevnik (2000): Yes, for any point in a Stein manifold.
- DD, Forstnerič (2007): Yes, for any point in a $(n-1)$-convex complex space.

A Riemann-Hilbert problem, a model case

First we present the simplest version in $\mathbb{C}^2$:

We are given the closed analytic disc $f(z) = (z, 0)$ in $\mathbb{C}^2$ and the holomorphically varying family of closed analytic discs around the boundary $f(b\mathbb{D})$: a map $g\colon b\mathbb{D} \times \mathbb{D} \to \mathbb{C}^2$ defined by $g(z, w) = (z, w)$.

We seek a closed analytic disc $h\colon \overline{\mathbb{D}} \to \mathbb{C}^2$ such that

- $h(z)$ lies near the set $g(z, b\mathbb{D})$, for each $z \in b\mathbb{D}$,
- $h(z)$ is close to $f(z)$ on a given compact subset of $\mathbb{D}$,
- h is not far from g outside the given compact subset of $\mathbb{D}$,
- $h(0) = f(0)$.

We may take $h(z) = (z, z^n)$ for n large.

A Riemann-Hilbert problem, general situation

Existence of the approximate solution of the Riemann-Hilbert boundary value problem is provided by:

Theorem

Let $f\colon \overline{\mathbb{D}} \to \mathbb{C}^n$ *be a closed analytic disc, and let* $g\colon b\mathbb{D} \times \overline{\mathbb{D}} \to \mathbb{C}^n$ *be a continuous map, such that for each* $z \in b\mathbb{D}$ *the map* $w \mapsto g(z,w)$ *is a closed analytic disc with the center* $f(z)$*.*
Given $\epsilon > 0$ *and* $r \in (0,1)$*, there exist* $r' \in [r,1)$ *and a closed analytic disc* h *in* $\mathbb{C}^n$ *with center* $f(0)$ *such that*

(i) $\mathrm{dist}(h(z), g(z, b\mathbb{D})) < \epsilon$*, for each* $z \in b\mathbb{D}$*,*

(ii) $\mathrm{dist}(h(\rho z), g(z, \overline{\mathbb{D}})) < \epsilon$*, for each* $\rho \in [r', 1], z \in b\mathbb{D}$*,*

(iii) $|h(z) - f(z)| < \epsilon$*, for each* z*,* $|z| < r'$*.*

A Riemann-Hilbert problem, formulation of the problem

The exact solution of the Riemann-Hilbert boundary value problem is a much deeper result:

A *Hilbert boundary value problem* is to find a closed analytic disc f in $\mathbb{C}$ such that its boundary value point $f(z)$, $z \in b\mathbb{D}$, lies in a prescribed curve $\gamma_z \subset \mathbb{C}$. The special case where γ_z is an affine real line was mentioned by Riemann in 1852 and solved by Hilbert in 1905.

A *nonlinear Riemann-Hilbert boundary value problem* is the following: given a smooth family of Jordan curves γ_z in $\mathbb{C}$ $(z \in b\mathbb{D})$ which all contain 0 in their interior, find a closed analytic disc f in $\mathbb{C}$ such that at each boundary point $f(z) \in \gamma_z$ $(z \in b\mathbb{D})$. It has been used in the study of polynomial hulls in $\mathbb{C}^2$.

The approximate solution of the Riemann-Hilbert boundary value problem has been used in constructions of proper holomorphic discs, in Poletsky theory, and lately in minimal surface theory.

A Riemann-Hilbert problem, general situation

Theorem

Let $f\colon \overline{\mathbb{D}} \to \mathbb{C}^n$ *be a closed analytic disc, and let* $g\colon b\mathbb{D} \times \overline{\mathbb{D}} \to \mathbb{C}^n$ *be a continuous map, such that for each* $z \in b\mathbb{D}$ *the map* $w \mapsto g(z,w)$ *is a closed analytic disc with the center* $f(z)$*.*
Given $\epsilon > 0$ *and* $r \in (0,1)$*, there exist* $r' \in [r,1)$ *and a closed analytic disc* h *in* $\mathbb{C}^n$ *with center* $f(0)$ *such that*

(i) $\mathrm{dist}(h(z), g(z, b\mathbb{D})) < \epsilon$, *for each* $z \in b\mathbb{D}$,

(ii) $\mathrm{dist}(h(\rho z), g(z, \overline{\mathbb{D}})) < \epsilon$, *for each* $\rho \in [r', 1], z \in b\mathbb{D}$,

(iii) $|h(z) - f(z)| < \epsilon$, *for each* z, $|z| < r'$.

A Riemann-Hilbert problem, idea of the proof

Write

$$g(z,w) = f(z) + \lambda(z,w), \qquad (z,w) \in b\mathbb{D} \times \overline{\mathbb{D}},$$

where λ is continuous on $b\mathbb{D} \times \overline{\mathbb{D}}$, and for every fixed $z \in b\mathbb{D}$ the function $w \mapsto \lambda(z,w)$ is a closed analytic disc centered at 0.

Claim: We can approximate λ uniformly on $b\mathbb{D} \times \overline{\mathbb{D}}$ by Laurent polynomials of the form

$$\widetilde{\lambda}(z,w) = \frac{1}{z^m} \sum_{j=1}^{N} A_j(z) w^j$$

with polynomial coefficients $A_j(z)$.

Then the map $h(z) = f(z) + \widetilde{\lambda}(z, z^k)$ for k large enough satisfies (i)-(iii) in the theorem.

A Riemann-Hilbert problem, proof of the Claim

Write

$$\frac{1}{\zeta - w} = \frac{1 - \frac{w^{n+1}}{\zeta^{n+1}} + \frac{w^{n+1}}{\zeta^{n+1}}}{\zeta(1 - \frac{w}{\zeta})} = \frac{1}{\zeta}\left(1 + \frac{w}{\zeta} + \frac{w^2}{\zeta^2} + \cdots + \frac{w^n}{\zeta^n}\right) + \frac{w^{n+1}}{\zeta^{n+1}(\zeta - w)}.$$

We get for $|w| < r' < 1$ by Cauchy formula

$$\begin{aligned} 2\pi i\,\lambda(z,w) &= \int_{b\mathbb{D}} \frac{\lambda(z,\zeta)}{\zeta - w} d\zeta \\ &= \int_{b\mathbb{D}} \frac{\lambda(z,\zeta)}{\zeta}\, d\zeta + \int_{b\mathbb{D}} \lambda(z,\zeta)\left(\frac{w}{\zeta^2} + \cdots + \frac{w^n}{\zeta^{n+1}}\right) d\zeta + \int_{b\mathbb{D}} \frac{w^{n+1}\lambda(z,\zeta)}{\zeta^{n+1}(\zeta - w)} d\zeta \\ &= \sum_{j=1}^{n} B_j(z) w^j + w^{n+1} \int_{b\mathbb{D}} \frac{\lambda(z,\zeta)}{\zeta^{n+1}(\zeta - w)} d\zeta. \end{aligned}$$

Since $|w| < r' < 1$, the last term is arbitrarily small, provided that n is large enough.

Functions B_j are continuous functions on $b\mathbb{D}$, so we can approximate $B_j(z)$, uniformly on $b\mathbb{D}$, arbitrarily well by $A_j(z)/z^m$ where A_j is a polynomial and where m can be chosen the same for all j. This proves the claim.

Constructions of proper holomorphic discs in $\mathbb{C}^n$

Idea of the construction:
Let $X = \cup_{j=1}^{\infty} X_j$, where $X_j \Subset X_{j+1} \Subset \mathbb{C}^n$. A proper holomorphic map $f\colon \mathbb{D} \to X$ is constructed inductively. At each inductive step we construct a closed analytic disc $f_j\colon \overline{\mathbb{D}} \to X$ with the following properties:

- the boundary of the analytic disc f_j lies outside X_j, i.e. $f_j(b\mathbb{D}) \subset X \setminus X_j$,
- f_j approximates f_{j-1} on a compact subset of $\mathbb{D}$,
- $f_{j-1}(0) = f_j(0)$.

If X is a convex domain then X can be written as a union of strictly convex domains X_j. Then the continuously varying family of analytic discs for the Riemann-Hilbert problem can be constructed by taking a paramatrized family of linear discs which are tangent to the boundary, thus the solution of the Riemann-Hilbert problem gives the inductive step.

For more general domains we need some convexity property of the boundary.

Constructions of proper holomorphic discs in general

In the case where X is a complex manifold, we have to assume that X has an exhaustion function with certain convexity properties. An *exhastion function* of a complex manifold X is a proper map $\phi\colon X \to \mathbb{R}$. If X is a convex domain in $\mathbb{C}^n$, the sets X_j are sublevel sets of a strictly convex exhaustion function.

Two new difficulties arise:

- Exhaustion function may have critical points. We change the exhaustion function slightly near the critical point and at the same time we preserve the convexity property.
- The Riemann-Hilbert problem works in Euclidean space, so it only provides corrections in the local coordinates. We need to globalize the local corrections using *holomorphic sprays*.

Holomorphic sprays

A holomorphic spray is a holomorphically varying family of holomorphic maps, more precisely:

Definition

Assume that D is a relatively compact domain with $\mathcal{C}^2$ boundary in $\mathbb{C}$. A *spray of maps* is a map $f\colon P \times \bar{D} \to X$, where P (the *parameter set* of the spray) is an open subset of a Euclidean space $\mathbb{C}^m$ containing the origin, such that the following hold:

(i) f is holomorphic on $P \times D$ and continuous on $P \times \bar{D}$,

(ii) the maps $f(0, \cdot\,)$, and $f(t, \cdot\,)$ agree at 0 for all $t \in P$, and

(iii) for $z \in \bar{D} \backslash \{0\}$ and $t \in P$ then

$$\partial_t f(t, z)\colon T_t\mathbb{C}^m \to T_{f(t,z)}X$$

is surjective (the *domination property*).

Holomorphic sprays

Holomorphic sprays were introduced by Gromov in 1989. He used sprays in Oka-Grauert theory:
The *Oka principle* first appeared in 1939 when Oka showed that a holomorphic line bundle on a domain of holomorphy is holomorphically trivial if (and only if) it is topologically trivial. Grauert extended Oka's theorem to principal fiber bundles with arbitrary complex Lie group fibers over Stein spaces.
Holomorphic sprays provide a very flexible tool and Gromov applied them to prove Oka principle in the more general context of sections of holomorphic submersions over Stein manifolds.

Existence of sprays

Lemma

Assume that D is a relatively compact domain with $\mathcal{C}^2$ boundary in $\mathbb{C}$. Given a continuous map $f_0 \colon \bar{D} \to X$, holomorphic on D, there exists a spray $f \colon P \times \bar{D} \to X$, such that $f(0, \cdot\,) = f_0$.

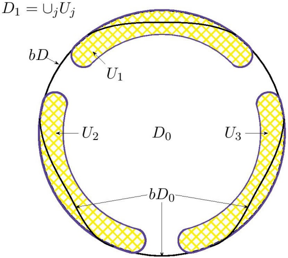
Cartan pair

Definition

A pair of open subsets $D_0, D_1 \Subset \mathbb{C}$ is said to be a *Cartan pair* if

(i) D_0, D_1, $D = D_0 \cup D_1$ and $D_{0,1} = D_0 \cap D_1$ are domains with smooth boundaries, and

(ii) $\overline{D_0 \setminus D_1} \cap \overline{D_1 \setminus D_0} = \emptyset$ (the separation property).



Gluing sprays

The following lemma actually explains how we get new sprays.

Lemma

Let (D_0, D_1) *be a Cartan pair in* $\mathbb{C}$ *such that* $0 \in D_0$. *Given a spray* $f\colon P_0 \times \bar{D}_0 \to X$ *there is an open set* $P \Subset P_0 \subset \mathbb{C}^m$ *containing* $0 \in \mathbb{C}^m$ *and satisfying the following. For every spray* $\tilde{f}\colon P_0 \times \bar{D}_1 \to X$ *such that* $\tilde{f}$ *is sufficiently uniformly close to* f *on* $P_0 \times \bar{D}_{0,1}$ *there exists a spray* $F\colon P \times \bar{D} \to X$ *enjoying the following properties:*

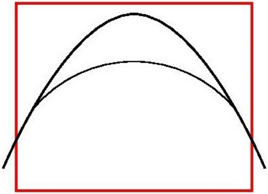
(i) *the restriction* $F\colon P \times \bar{D}_0 \to X$ *is uniformly close to* $f\colon P \times \bar{D}_0 \to X$,

(ii) $F(0,0) = f(0,0)$,

(iii) $F(t,z) \in \{\tilde{f}(s,z)\colon s \in P_0\}$ *for each* $t \in P$ *and* $z \in \bar{D}_1$.

Constructions of proper holomorphic discs in general

We build the manifold X (outside critical level sets) by attaching convex bumps, i.e. $X = \cup X_j$, where $X_j \setminus X_{j-1}$ is a convex bump in some local coordinates:



In the inductive construction of a proper map we start with a small closed analytic disc centered at a given point. We embedd this disc into a spray of discs. We construct inductively a sequence of sprays of closed analytic discs such that at each step boundaries of all the discs in the spray lie outside X_j.

Related results

In the past few years both methods have been applied in minimal surface theory:

- Alarcón, Forstnerič (2013): Every bordered Riemann surface admits a complete proper holomorphic immersion to the unit ball of $\mathbb{C}^2$, and a complete proper holomorphic embedding to the unit ball of $\mathbb{C}^3$.
- Alarcón, Forstnerič (2015): Every bordered Riemann surface carries a conformal complete minimal immersion into $\mathbb{R}^3$ with bounded image.
- Alarcón, DD, Forstnerič, López (2015): Let $D \subset \mathbb{R}^n$ ($n \geq 3$) be a bounded strictly convex domain with $\mathcal{C}^2$ smooth boundary, let M be a compact bordered Riemann surface, and let $F\colon M \to D$ be a conformal minimal immersion of class $\mathcal{C}^1$. If $F(M) \subset D$, then F can be approximated uniformly on compacts in $M \setminus bM$ by continuous maps $G\colon M \to D$ such that $G\colon M \setminus bM \to D$ is a conformal complete proper minimal immersion.

Related results

- Alarcón, Forstnerič, López (2016): Let M be a compact bordered Riemann surface with nonempty boundary $bM \subset M$. Every smooth Legendrian curve $F\colon M \to \mathbb{C}^{2n+1}$, holomorphic in $M \setminus bM$, can be approximated uniformly on M by continuous injective maps $G\colon M \to \mathbb{C}^{2n+1}$ whose restriction to the interior $M \setminus bM$ is a complete holomorphic Legendrian embedding.