Condensing KKM maps and its applications

Ivan D. Aranđelović, Z.Mitrović

October 16, 2015

Zoran Mitrović and Ivan Aranđelović.

Existence of Generalized Best Approximations, Journal of Nonlinear and Convex Analysis, 15:4 (2014), 787–792.

In this paper we gave further extension of the best approximation theorems obtained by Ky Fan, J. Prolla and A. Carbone. In our result conditions of almost-affinity, quasi-convexity and compactness are omitting.

K. Fan, Extensions of two fixed point theorems of F.E. Browder, Math. Z. 112 (1969), 234–240.

J. B. Prolla, Fixed point theorems for set-valued mappings and existence of best approximations, Numer. Funct. Anal. Optim. 5(1983), 449–455.

A. Carbone, A note on a theorem of Prolla, Indian J. Pure. Appl. Math. 23 (1991), 257–260.

A. Carbone, An application of KKM-map principle, Int. J. Math. Math. Sci. 15 No. 4 (1992) 659–662.

The theory of measures of non-compactness has many applications in Topology, Functional analysis and Operator theory. There are many nonequivalent definitions of this notion on metric and topological spaces. First of them was introduced by Kuratowski in 1930. In this paper we shall used definition of L. Pasicki.

L. Pasicki, On the measure of non-compactness, Comment. Math. Prace Mat. 21 (1979), 203–205.

Definition. *Let X be a metric space. Measure of non-compactness on X is an arbitrary function $\phi : \mathcal{P}(X) \to [0, \infty]$ which satisfies following conditions:*

1) $\phi(A) = 0$ if and only if A is totally bounded set;

2) from $A \subseteq B$ follows $\phi(A) \leq \phi(B)$.

5) for each $A \subseteq X$ and $x \in X$ $\phi(A \cup \{x\}) = \phi(A)$.

Theorem. *(L. Pasicki) Let X be a complete metric space and ϕ measure of non-compactness on X. If $\{B_n\}_{n\in N}$ is a sequence of its nonempty closed subsets such that:*

1) $B_{n+1} \subseteq B_n$ *for any* $n \in N$;

2) $\lim_{n\to\infty} \phi(B_n) = 0$;

then $K = \bigcap_{n\in N} B_n$ is a nonempty, compact set.

The most important examples of measures of non-compactness on a metric space (X, d) are:

1) Kuratowski's measure

$\alpha(A) = \text{inf}\{r > 0 :$

$A \subseteq \bigcup_{i=1}^{n} S_i, S_i \subseteq X, \text{diam}(S_i) < r, 1 \leq i \leq n\}$

2) Hausdorff's measure $\chi(A) = \text{inf}\{\varepsilon > 0 \ : \ A$ has a finite $\varepsilon-$ net in $X\}$;

3) measure of Istratescu $I(A) = \text{inf}\{\varepsilon > 0 \ : \ A$ contains no infinite $\varepsilon-$ discrete set in $A\}$.

Relations between this functions are given by following inequality, which are obtained by Danes:

$$\chi(A) \leq I(A) \leq \alpha(A) \leq 2\chi(A).$$

J.Danes, On the Istratescu's measure of non-compactness, Bull. Math. Soc. R. S. Roumanie 16 (64) (1972), 403–406.

The famous Brouwer fixed point theorem (any continuous function $f : K \to K$ has at least one fixed point, where $K \subseteq R^n$ is non-empty, compact and convex set of R^n) for $n = 3$ was proved by him in 1909; equivalent results were established earlier by Henri Poincare in 1883 and P. Bohl in 1904. It was Hadamard who in 1910 gave (using the Kronecker index) the first proof for an arbitrary n. In 1912 Brouwer gave another proof using the simplicial approximation technique, and notions of degree. In 1927 Schauder obtained first infinite dimensional generalization of Brouwer fixed point theorem and gave its applications in the theory of elliptic equations of Mathematical Physic. In 1928 J. von Neumann by using Brouwer theorem proved existence of solution of matrix zero sums games.

A short and simple proof of Brouwer theorem was given in 1929 by Knaster, Kuratowski and Mazurkiewicz. This proof is based on one corollary of the Sperner's lemma which is known as KKM lemma. First infinite dimensional generalization of this statement (so called KKM principle) was obtained by Ky Fan in 1961. This statement, which is an infinite dimensional generalization of classical KKM lemma, is known as KKM principle.

K. Fan, A generalization of Tychonoff's fixed point Theorem, Math. Ann. 142, (1961), 305–310.

Fixed point formulation of Fans result, which is also very applicable, so-called Fan-Browders theorem was obtained by Felix Browder in 1968.

Let X and Y be non-empty sets; we denote by 2^X a family of all non-empty subsets of X, $\mathcal{F}(X)$ a family of all non-empty finite subsets of X and $\mathcal{P}(X)$ a family of all subsets of X. A multi-function G from X into Y is a map $G : X \to 2^Y$. If $G : X \to 2^Y$ we define

$G^{-1}, G^* : Y \to \mathcal{P}(X)$ and $G^c : X \to \mathcal{P}(Y)$ by $G^{-1}(y) = \{x \in X : y \in G(x)\}$, $G^*(y) = \{x \in X : y \notin G(x)\}$ and $G^c(x) = Y \backslash G(x)$.

Definition. *Let C be a nonempty subset of a topological vector space X. A map $G : C \to 2^X$ is called KKM map if for every $D \in \mathcal{F}(C)$ we have*

$$\text{conv}(D) \subseteq \bigcup_{x \in D} G(x).$$

Next statement is Ky Fan's KKM principle.

Theorem. *Let X be a topological vector space, K be a nonempty subset of X and $G : K \to 2^X$ a KKM map with closed values. If $G(x)$ is compact for at least one $x \in K$ then $\bigcap_{x \in K} G(x) \neq \emptyset$.*

Definition. *Let E a metric linear space, ϕ measure of non-compactness on E, and $X \subseteq E$. A multi-function $G : X \to 2^E$ is condensing multi-function if for every $\varepsilon > 0$ there exist $n \in \mathcal{N}$ and $x_1, ..., x_n \in X$ such that*

$$\phi(G(x_1) \bigcap ... \bigcap G(x_n)) < \varepsilon.$$

A condensing multi function $G : X \to 2^E$ is condensing KKM multi-function if it is KKM multi-function.

The following result, generalize recent results of C. Horvath.

C. Horvath, Point fixes et coincidences pour les applications multivoques sans convexité, C.R. Acad. Sci. Paris, Serie I 296 (1983), 403–406.

C. Horvath, *Measure of Non-compactness and Multivalued Mappings in Complete Metric Topological Vector Space, J. Math. Anal. Appl, 108*(1985), 403–408.

Theorem. *Let E be a complete metric linear space, ϕ measure of non-compactness on E, $X \in 2^E$ and let $G : X \to 2^E$ be a condensing KKM multi-function. If $G(x)$ is a closed set for each $x \in X$, then $\bigcap_{x \in X} G(x)$ is non-empty and compact set.*

Lemma. *(*) Let* E *be a topological space space, and* G : $X \longrightarrow 2^X$ *be a multi-function. If:*

1) $x \in G(x)$ *for each* $x \in X$*;*

2) $G^{*-1}(x)$ *is a convex set for each* $x \in X$*;*

G *is a KKM multifunction.*

We prove that: if G is not KKM multi-function, then G^* has a fixed point. If G is not KKM multi-function, than there exists a finite subset $\{x_1, ..., x_n\} \in \mathcal{F}(X)$ and $x_0 \in X$ such that $x_0 \in \text{conv}(\{x_1, ..., x_n\})$ and $x_0 \notin \bigcup_{i=1}^n G(x_i)$. This implies $x_0 \in \bigcap_{i=1}^n G^*(x_i)$ and hence $x_i \in G^{*-1}(x_0)$ for $1 \leq i \leq n$. Since $G^{*-1}(x_0)$ is a convex set, we have

$$\text{conv}(\{x_1, ..., x_n\}) \subseteq G^{*-1}(x_0), \text{ and so } x_0 \in G^*(x_0).$$

Corollary. *(*) Let E be a topological vector space, and $G : X \longrightarrow 2^X$ be a multi-function, such that $G^{*-1}(x)$ is convex set for any $x \in X$. Then the following statements are equivalent:*

1) $x \in G(x)$ for any $x \in X$;

2) G is KKM multi-function.

2) follows 1) by lemma (*). If G KKM multi-function, then for each x we have $H(\{x\}) \subseteq G(x)$, which implies $x \in G(x)$ for any $x \in X$.

Replacement of G^* with G in Corollary (*) implies

Corollary. *(**) Let E be a topological vector space, and $G : X \longrightarrow 2^X$ be a multi-function, such that $G^{-1}(x)$ is convex set for any $x \in X$. Then the following statements are equivalent:*

1) $x \in G(x)$ za neko $x \in X$;

2) G^* is not KKM multi-function.

Lemma. *(**) Let* E *be a complete metric linear space,* ϕ *measure of non-compactness on* E *and* $X \in 2^E$ *convex set, and let* $T : X \longrightarrow \mathcal{P}(X)$ *be multi-functions such that:*

1) for each $x \in X$, $T(x)$ *is nonempty and convex set;*

2) for each $y \in X$, $T^{-1}(y)$ *is open set;*

3) for each $t > 0$ *there exists* $x \in X$ *such that* $\phi(T^{-1*}(x) < t$;

then T *has a fixed point.*

We shall proved that T^{-1} has a fixed point. By Lemma (*) for $G = T^{-1*}$ we need only to show that G is not KKM multi-function and set $G^{*-1}(y)$ is convex for each $y \in X$ (then statement $x \in G(x)$ is not true for each $x \in X$, which implies that there exists $x_0 \in X$ such that $x_0 \in G^*(x_0) = T^{-1}(x_0)$). 2) is satisfied because $G^{*-1}(y) = T(y)$ for any $y \in X$. If G is KKM multi-function, then by Lemma (*) $\bigcap_{x \in X} G(x) \neq \emptyset$ because $G(x)$ are closed (which implies compact) set for for each $x \in X$. Hence we obtained $\bigcup_{x \in X} G^*(x) \neq X$, which is contradiction, because by 1) we have that $T(x) \neq \emptyset$ for any $x \in X$.

In this talk we present the following fixed point result for multi-functions, which generalize famous Fan - Browders theorem for metrizable spaces.

Theorem. *Let E be a complete metric linear space, ϕ measure of non-compactness on E and $X \in 2^E$ convex set, and let $S, T : X \longrightarrow \mathcal{P}(X)$ be two multi-functions such that:*

1) $S(x)$ is nonempty set and $S(x) \subseteq T(x)$ for each $x \in X$;

2) $T(x)$ is convex set for each $x \in X$;

3) $S^{-1}(y)$ is open set for each $y \in X$.

If for any $t > 0$ there exists $x_t \in X$ such that $\phi(S^{-1*}(x_t)) < t$, then T has a fixed point.

Its enough to prove that T^{-1} has a fixed point. $\bigcap_{y \in X} S^{-1*}(y) = \emptyset$ because $S(x) \neq \emptyset$ for any $x \in X$. For each $y \in X$ $S^{-1*}(y)$ is closed set, which implies that S^{-1*} is not KKM multi-function. By Corollary (**) we obtained that T^{-1*} is not KKM multi-function, because $T^{-1*}(x) \subseteq S^{-1*}(x)$ za sve $x \in X$. By 1) we have that for each $x \in X$ set $(T^{-1*})^{-1*}(x) = T(x)$ is convex, which implies, by Lemma (*), that there exists $x \in X$ such that $x \in (T^{-1*})^{*}(x) = T^{-1}(x)$.

In 1930's, earlier works of Nikodym, Mazur, Schauder initiated the abstract aproach to problems in calculus of variations. For further development of this theory the most important results are existence of solutions of the nonlinear variational inequalities obtained by Hartman and Stamppacia in 1966 and minimax inequality presented by Ky Fan in 1972. Its result has been used ina large variety of problems in nonlinear analysis, convex analysis, partial differential equations, mechanics, physics, optimization and control theory.

As application we obtained following variational inequality which generalizes famous results of of Yen and Ky Fan for metrizable spaces.

Corollary. *Let E be a complete metric linear space, ϕ measure of non-compactness on E and $X \in 2^E$ convex set, and let $p, q : X^2 \to \mathcal{R}$ be two real functions such that:*

1) $p(x, y) \leq q(x, y)$ for all $x, y \in X$;

2) function $x \longrightarrow q(x, y)$ is quasi concave on X for each $y \in X$;

3) function $y \longrightarrow p(x, y)$ is lower semi continuous for each $x \in X$;

4) for any $t > 0$ there exists $x_t \in X$ such that

$$\phi(\{y \in X : p(x_t, y) \leq \sup_{x \in X} q(x, x)\}) < t.$$

Then there exists $\widehat{y} \in X$ such that:

$$\sup_{x \in X} p(x, \widehat{y}) \leq \sup_{x \in X} q(x, x).$$

Let $\lambda = sup_{x \in X} q(x, x)$ and let $S, T : C \longrightarrow \mathcal{P}(X)$ be two set-valued maps defined by:

$S(y) = \{x \in X : p(x, y) > \lambda\}$,

$T(x) = \{y \in X : q(x, y) > \lambda\}$.

By assumptions we get that:

a) T has not a fixed point;

b) $S(y) \subseteq T(y)$ for any $y \in X$;

c) for each $y \in X$, $T(y)$ is convex set;

d) for each $x \in X$, $S^{-1}(x) = \{y \in X : h(x, y) > \lambda\}$ is open subset of X;

e) for each $t > 0$ there exists $x_t \in X$ such that $\phi(S^{-1*}(x_t)) < t$.

From Theorem, it follows, that there exists $\widehat{y} \in X$ such that $S(\widehat{y}) = \emptyset$, which imlies $p(x, \widehat{y}) \leq \lambda\}$ for any $x \in X$.